

PADMASHREE KRUTARTHA ACHARYA INSTITUTE OF ENGINEERING & TECHNOLOGY, BARGARH



3rd Semester, Electrical Engineering

Engineering Mathematics-III (TH-1)

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① Complex Number

$i = \sqrt{-1}$ is called an imaginary number

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{(i^2)^2} = \frac{i}{(-1)^2} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{(i^2)^2} = \frac{1}{(-1)^2} = 1$$

$$i^{4n} = (i^4)^n = 1^n = 1$$

$$i^{4n+1} = i^{4n} \cdot i = 1 \cdot i = i$$

$$i^{4n+2} = i^{4n} \cdot i^2 = 1 \cdot (-1) = -1$$

$$i^{4n+3} = i^{4n} \cdot i^3 = 1 \cdot (-i) = -i$$

$$i^{-4n} = \frac{1}{i^{4n}} = \frac{1}{1} = 1$$

$$i^{-(4n+1)} = \frac{1}{i^{4n+1}} = \frac{1}{i} = -i$$

$$i^{-(4n+2)} = \frac{1}{i^{4n+2}} = \frac{1}{-1} = -1$$

$$i^{-(4n+3)} = \frac{1}{i^{4n+3}} = \frac{1}{-i} = i$$

Def.ⁿ The number z which is written in the form

$$z = x + iy, \quad x, y \in \mathbb{R}$$

is called a complex number and the numbers x and y are called the real and imaginary parts of z .

$$\text{i.e., } \text{Re}(z) = x \text{ and } \text{Im}(z) = y$$

Equality of Complex Number

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$

Addition and Subtraction of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers then $z_1 + z_2 = (a + c) + i(b + d)$

$$\text{and } z_1 - z_2 = (a - c) + i(b - d)$$

Multiplication of Complex Numbers

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

2) Division of Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, $z_2 \neq 0$ then $\frac{z_1}{z_2} = \frac{a + ib}{c + id}$

$$= \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + ibc - iad - i^2bd}{c^2 - i^2d^2}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}$$

Q. Express in the form of $x + iy$

1) $\frac{2 + 3i}{5 - 2i}$

2) $\frac{3 + 5i}{2 - 3i}$

3) $\frac{(1 + i)(2 + i)}{3 + i}$

Q. Find the additive inverse and the multiplicative inverse of 1) $2 - 3i$ 2) $-3 + 4i$

Conjugate of a complex Number

If $z = a + ib$ is a complex number, then the conjugate of z is denoted by

$$\bar{z} = a - ib$$

1) If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$

2) If $z = 5 + 7i$ then $\bar{z} = 5 - 7i$

3) If $z = 2i - 5$ then $\bar{z} = -5 - 2i$

1) $\overline{\bar{z}} = z$ Let $z = a + ib$
 $\bar{z} = a - ib$
 $\overline{\bar{z}} = a + ib = z$

2) $z + \bar{z} = 2 \operatorname{Re}(z)$ Let $z = a + ib$
 $\bar{z} = a - ib$
L.H.S. $= z + \bar{z}$
 $= a + ib + a - ib$
 $= 2a = 2 \operatorname{Re}(z)$
 $= \text{R.H.S.}$

$$3) \quad x - \bar{x} = 2 \operatorname{Im}(x)$$

$$4) \quad \overline{x_1 + x_2} = \bar{x}_1 + \bar{x}_2$$

$$5) \quad \overline{x_1 - x_2} = \bar{x}_1 - \bar{x}_2 \quad 6) \quad \overline{x_1 x_2} = \bar{x}_1 \bar{x}_2$$

$$\text{let } x_1 = a + ib \quad \text{and} \quad x_2 = c + id$$

$$\bar{x}_1 = a - ib \quad \bar{x}_2 = c - id$$

$$7) \quad \left(\frac{\bar{x}_1}{x_2} \right) = \frac{\bar{x}_1}{x_2}, \quad x_2 \neq 0$$

$$\begin{aligned} x_1 + x_2 &= a + ib + c + id \\ &= (a+c) + i(b+d) \\ \overline{x_1 + x_2} &= (a+c) - i(b+d) \\ &= (a-ib) + (c-id) \\ &= \bar{x}_1 + \bar{x}_2 \end{aligned}$$

$$\begin{aligned} 5) \quad x_1 - x_2 &= (a+ib) - (c+id) \\ &= (a-c) + i(b-d) \\ \overline{x_1 - x_2} &= (a-c) - i(b-d) \\ &= (a-ib) - c + id \\ &= (a-ib) - (c-id) \\ &= \bar{x}_1 - \bar{x}_2 \end{aligned}$$

$$\begin{aligned} &= \frac{(a-ib)(c+id)}{(c+id)(c-id)} \\ &= \frac{a-ib}{c-id} = \frac{\bar{x}_1}{x_2} \end{aligned}$$

$$\begin{aligned} 6) \quad x_1 x_2 &= (a+ib)(c+id) \\ &= ac + ibc + iad + i^2 bd \\ &= (ac - bd) + i(bc + ad) \\ \overline{x_1 x_2} &= (ac - bd) - i(bc + ad) \\ &= ac - ibc - iad + i^2 bd \\ &= c(a-ib) - id(a-ib) \\ &= (a-ib)(c-id) \\ &= \bar{x}_1 \bar{x}_2 \end{aligned}$$

$$\begin{aligned} 7) \quad \frac{x_1}{x_2} &= \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac + ibc - iad - i^2 bd}{c^2 - i^2 d^2} \\ &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2} \end{aligned}$$

$$\begin{aligned} \left(\frac{x_1}{x_2} \right) &= \frac{ac + bd}{c^2 + d^2} - i \frac{bc - ad}{c^2 + d^2} = \frac{ac + bd - ibc + iad}{c^2 + d^2} \\ &= \frac{ac - ibc + iad - i^2 bd}{c^2 - i^2 d^2} = \frac{c(a-ib) + id(a-ib)}{(c+id)(c-id)} \end{aligned}$$

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Modulus of a Complex Numbers

If $z = a + ib$ is a complex number, then the modulus of z is denoted by

$$|z| = \sqrt{a^2 + b^2}$$

6) Find $|z|$ if $z = \frac{1+2i}{1-3i}$

Q. Find the modulus of 1) $2 - 3i$
2) $5 + \sqrt{2}i$ 3) $3 + 4i$ 4) $2\sqrt{6} - i$ 5) $-\sqrt{15} - \sqrt{10}i$

Sol. 1) Let $z = 2 - 3i$ then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$

2) Let $z = 5 + \sqrt{2}i$ then $|z| = \sqrt{5^2 + (\sqrt{2})^2} = \sqrt{25 + 2} = \sqrt{27}$

3) Let $z = 3 + 4i$ then $|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

4) Let $z = 2\sqrt{6} - i$ then $|z| = \sqrt{(2\sqrt{6})^2 + (-1)^2} = \sqrt{4 \times 6 + 1} = \sqrt{25} = 5$

5) Let $z = -\sqrt{15} - \sqrt{10}i$ then $|z| = \sqrt{(-\sqrt{15})^2 + (-\sqrt{10})^2} = \sqrt{15 + 10} = \sqrt{25} = 5$

Q. If z is a complex number then 1) $z\bar{z} = |z|^2$
2) $|z| = |\bar{z}|$ 3) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Sol. Let $z = a + ib$
 $\bar{z} = a - ib$ and $|z| = \sqrt{a^2 + b^2}$

1) L.H.S. $= z\bar{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2 = R.H.S.$

2) $|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$ 3) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

Q. If z_1 and z_2 are two complex numbers then

1) $|z_1 z_2| = |z_1| |z_2|$ 2) $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$

Proof 1) Let $z_1 = a + ib$ and $z_2 = c + id$

$$|z_1| = \sqrt{a^2 + b^2} \quad |z_2| = \sqrt{c^2 + d^2}$$

~~Let~~ $z_1 z_2 = (a + ib)(c + id)$

$$= ac + ibc + iad + i^2 bd$$

$$= (ac - bd) + i(bc + ad)$$

$$|z_1 z_2| = \sqrt{(ac - bd)^2 + (bc + ad)^2}$$

$$= \sqrt{a^2 c^2 + b^2 d^2 - 2abcd + b^2 c^2 + a^2 d^2 + 2abcd}$$

$$= \sqrt{a^2 c^2 + b^2 c^2 + a^2 d^2 + b^2 d^2}$$

$$= \sqrt{c^2(a^2 + b^2) + d^2(a^2 + b^2)}$$

$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$$

$$= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z_1| |z_2|$$

2) $\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)}$

$$= \frac{ac + ibc - iad - i^2 bd}{c^2 - i^2 d^2}$$

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}$$

$$\left| \frac{z_1}{z_2} \right| = \sqrt{\left(\frac{ac + bd}{c^2 + d^2} \right)^2 + \left(\frac{bc - ad}{c^2 + d^2} \right)^2}$$

$$= \sqrt{\frac{a^2 c^2 + b^2 d^2 + 2abcd + b^2 c^2 + a^2 d^2 - 2abcd}{(c^2 + d^2)^2}}$$

~~or~~
$$= \sqrt{\frac{a^2 c^2 + a^2 d^2 + b^2 d^2 + b^2 c^2}{(c^2 + d^2)^2}}$$

$$= \sqrt{\frac{a^2(c^2 + d^2) + b^2(c^2 + d^2)}{(c^2 + d^2)^2}} = \sqrt{\frac{a^2 + b^2}{c^2 + d^2}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

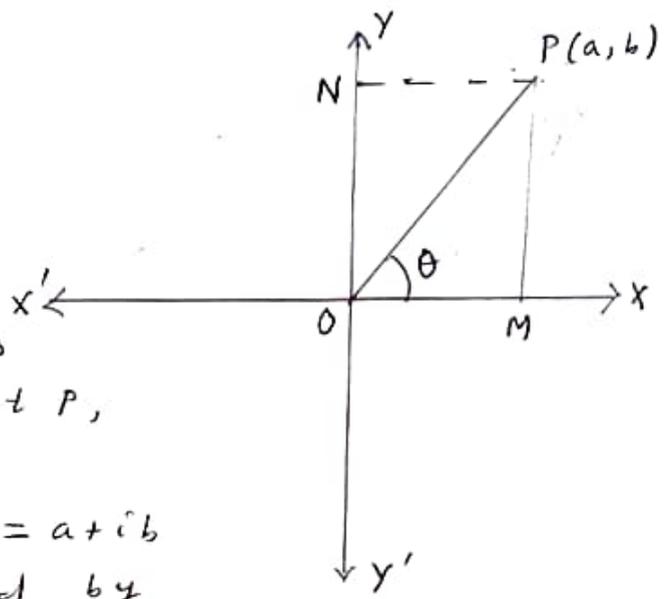
$$= \frac{|z_1|}{|z_2|}$$

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Geometrical representation of Complex Numbers

We can represent a complex number $z = a + ib$ geometrically with the help of a rectangular system of axes OX and OY . It is clear that any point P in the XOY plane is uniquely determined if we know its distance from OX and from OY .

If a point P is given and $PN = a$ and $PM = b$ then P represents the point (a, b) . Similarly given a pair (a, b) of real numbers we can always plot the point P , which it represents.



The complex number $z = a + ib$ is also uniquely determined by the real numbers a, b . So we represent $z = a + ib$ by the point $P(a, b)$. The points $a = a + i0$ are represented by points on the real axis OX and the points $ib = 0 + ib$ are represented by points on the imaginary axis OY . When complex numbers are represented by points in the plane, the plane is called the complex plane or the Argand Plane or Gaussian plane.

The angle θ which OP makes with positive direction of X axis in anticlockwise direction is called argument or amplitude of z . It is denoted by $\arg(z)$ or $\text{amp}(z)$

Consider a complex number $z = a + ib$ ($a, b \in \mathbb{R}$)

In ΔPOM , $PN = OM = a$

$ON = PM = b$ and $OP = r$

$$\sin \theta = \frac{PM}{OP} = \frac{b}{r} \quad \text{and} \quad \cos \theta = \frac{OM}{OP} = \frac{a}{r}$$

$$\Rightarrow b = r \sin \theta \quad \Rightarrow a = r \cos \theta$$

$$\text{Then } z = a + ib = r \cos \theta + i r \sin \theta \quad \text{Polar Form}$$
$$= r (\cos \theta + i \sin \theta)$$

$$2) a^2 + b^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta \quad \text{Modulus of } z$$
$$= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$\Rightarrow r = +\sqrt{a^2 + b^2} \quad (\text{taking the +ve value of the square root})$$
$$= |z|$$

$$3) \frac{b}{a} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

Argument or Amplitude of z

$$\Rightarrow \theta = \tan^{-1} \frac{b}{a}$$

$$\therefore \text{ang}(z) \text{ or amp}(z) = \tan^{-1} \frac{b}{a}$$

5) Note a) The unique value of θ such that $-\pi < \theta < \pi$ is called the principal value of the amplitude or principal argument.

b) If $z = a + ib$ then

- i) $\text{Arg}(z) = \tan^{-1} \frac{b}{a}$ when $a > 0, b > 0$
- ii) $\text{Arg}(z) = \pi - \tan^{-1} \frac{b}{a}$ when $a < 0, b > 0$
- iii) $\text{Arg}(z) = -\pi + \tan^{-1} \frac{b}{a}$ when $a < 0, b < 0$
- iv) $\text{Arg}(z) = -\tan^{-1} \frac{b}{a} = 2\pi - \tan^{-1} \frac{b}{a}$ when $a > 0, b < 0$

c) For any two complex numbers z_1 and z_2

$$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$$

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

$$\text{Arg}(\bar{z}) = -\text{Arg}(z)$$

Q. Find the modulus and the argument of the complex numbers.

- 1) $1 + \sqrt{3}i$
- 2) $-2 + 2\sqrt{3}i$
- 3) $-\sqrt{3} - i$
- 4) $2\sqrt{3} - 2i$

Q. Write the polar form of the following complex number 1) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Sol. Here $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ $a < 0, b < 0$

$$|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow r = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$$

$$\text{Arg}(z) = -\pi + \tan^{-1} \frac{b}{a}$$

$$\Rightarrow \theta = -\pi + \tan^{-1} \left(\frac{-\frac{\sqrt{3}/2}{-1/2}}\right) = -\pi + \tan^{-1} \sqrt{3} = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$$

Hence in the polar form

$$z = r (\cos \theta + i \sin \theta)$$

$$= 1 \left(\cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3} \right)$$

Square Roots of a Complex Numbers

Find the square root of $a+ib$

Solⁿ let $\sqrt{a+ib} = x+iy$

$$\Rightarrow a+ib = x^2 + i^2 y^2 + i 2xy$$

$$= (x^2 - y^2) + i 2xy$$

$$\therefore x^2 - y^2 = a \text{ --- (i) and } 2xy = b$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2$$

$$= a^2 + b^2$$

$$\Rightarrow x^2 + y^2 = \sqrt{a^2 + b^2} \text{ --- (ii)}$$

Adding and subtracting (i) and (ii),

$$2x^2 = \sqrt{a^2 + b^2} + a$$

$$\text{and } 2y^2 = \sqrt{a^2 + b^2} - a$$

$$\Rightarrow x^2 = \frac{\sqrt{a^2 + b^2} + a}{2}$$

$$\Rightarrow y^2 = \frac{\sqrt{a^2 + b^2} - a}{2}$$

$$\Rightarrow x = \pm \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}}$$

$$\Rightarrow y = \pm \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

(i) If b is positive then x and y are of same sign, i.e.,

$$\sqrt{a+ib} = \pm \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} \pm i \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

(ii) If b is negative then x and y are of different sign, i.e.,

$$\sqrt{a+ib} = \pm \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} \mp i \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

Q. Find the square roots of 1) $3+4i$ 2) $-5+12\sqrt{-1}$
3) $-7-24i$ 4) $-15-8i$

Solⁿ 1) let $\sqrt{3+4i} = x+iy$

$$\Rightarrow 3+4i = x^2 + i^2 y^2 + i 2xy$$

$$= (x^2 - y^2) + i 2xy$$

6) $\therefore x^2 - y^2 = 3$ — (i) and $2xy = 4$

$$\begin{aligned}(x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 = 25\end{aligned}$$

$\Rightarrow x^2 + y^2 = 5$ — (ii)

Adding and subtracting (i) and (ii),

$$\begin{aligned}2x^2 &= 8 & \text{and} & \quad 2y^2 = 2 \\ \Rightarrow x^2 &= 4 & \Rightarrow y^2 &= 1 \\ \Rightarrow x &= \pm 2 & \Rightarrow y &= \pm 1\end{aligned}$$

$\therefore 2xy = 4$ is positive

$\therefore x$ and y are of same sign.

Hence $\sqrt{3+4i} = \pm(2+i)$

3) Let $\sqrt{-7-24i} = x+iy$

$$\begin{aligned}\Rightarrow -7-24i &= x^2 + i^2y^2 + i2xy \\ &= (x^2 - y^2) + i2xy\end{aligned}$$

$\therefore x^2 - y^2 = -7$ — (i) and $2xy = -24$

$$\begin{aligned}(x^2 + y^2)^2 &= (x^2 - y^2)^2 + 4x^2y^2 \\ &= (-7)^2 + (-24)^2 \\ &= \sqrt{49 + 576} = \sqrt{625} \quad \# \text{VVA}\end{aligned}$$

$\Rightarrow x^2 + y^2 = 25$ — (ii)

Adding and subtracting (i) and (ii),

$$\begin{aligned}2x^2 &= 18 & \text{and} & \quad 2y^2 = 32 \\ \Rightarrow x^2 &= 9 & \Rightarrow y^2 &= 16 \\ \Rightarrow x &= \pm 3 & \Rightarrow y &= \pm 4\end{aligned}$$

$\therefore 2xy = -24$ is negative

$\therefore x$ and y are opposite sign.

Hence $\sqrt{-7-24i} = 3-4i$ or $-3+4i$

Cube Roots of Unity

$$\text{Let } x = \sqrt[3]{1}$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x^3 - 1 = 0$$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\therefore x-1 = 0 \quad \text{or} \quad x^2+x+1 = 0$$

$$\Rightarrow x = 1$$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Thus cube roots of unity are $x = 1$, $\frac{-1 + \sqrt{3}i}{2}$, $\frac{-1 - \sqrt{3}i}{2}$

Clearly, one of the cube root of unity is real and the other two are complex number.

Properties of cube roots of unity

i) Cube roots of unity are $1, \omega, \omega^2$

$$\text{where } \omega = \frac{-1 + \sqrt{3}i}{2} \quad \text{and} \quad \omega^2 = \frac{-1 - \sqrt{3}i}{2}$$

ii) The two complex roots of the equation are conjugate of each other.

$$\text{Let } \omega = \frac{-1 + \sqrt{3}i}{2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\bar{\omega} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{-1 - \sqrt{3}i}{2} = \omega^2$$

$$\text{And } \omega^2 = \frac{-1 - \sqrt{3}i}{2} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\bar{\omega^2} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1 + \sqrt{3}i}{2} = \omega$$

iii) ~~The square of any cube root of unity is the other.~~

Each complex ~~cube~~ cube root of unity is the square of the other.

7) The complex cube roots of unity are

$$\frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \frac{-1 - i\sqrt{3}}{2}$$

$$\begin{aligned} \text{Now } \left(\frac{-1 + i\sqrt{3}}{2}\right)^2 &= \frac{1 + i^2 \cdot 3 - 2i\sqrt{3}}{4} = \frac{1 - 3 - 2\sqrt{3}i}{4} \\ &= \frac{-2 - 2\sqrt{3}i}{4} = \frac{-1 - \sqrt{3}i}{2} \end{aligned}$$

$$\begin{aligned} \text{And } \left(\frac{-1 - i\sqrt{3}}{2}\right)^2 &= \frac{1 + 3i^2 + 2\sqrt{3}i}{4} = \frac{1 - 3 + 2\sqrt{3}i}{4} \\ &= \frac{-2 + 2\sqrt{3}i}{4} = \frac{-1 + \sqrt{3}i}{2} \end{aligned}$$

~~Now $w^2 = \frac{-1 + i\sqrt{3}}{2}$ and $w = \frac{-1 - i\sqrt{3}}{2}$~~

iv) The sum of the three cube roots of unity is zero

$$\begin{aligned} \text{i.e., } 1 + \frac{-1 + i\sqrt{3}}{2} + \frac{-1 - i\sqrt{3}}{2} &= \frac{2 - 1 + i\sqrt{3} - 1 - i\sqrt{3}}{2} \\ &= \frac{0}{2} = 0 \end{aligned}$$

$$\Rightarrow 1 + w + w^2 = 0$$

v) The product of the three cube roots of unity is 1

$$\begin{aligned} \text{i.e., } 1 \cdot \left(\frac{-1 + i\sqrt{3}}{2}\right) \left(\frac{-1 - i\sqrt{3}}{2}\right) &= \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{4} \\ &= \frac{1 - i^2 \cdot 3}{4} = \frac{1 + 3}{4} = \frac{4}{4} = 1 \end{aligned}$$

$$\Rightarrow 1 \cdot w \cdot w^2 = 1$$

$$\Rightarrow w^3 = 1$$

vi) Each complex cube root of unity is the reciprocal of the other.

$$\text{We have, } w^3 = 1 \Rightarrow w \cdot w^2 = 1 \Rightarrow w = \frac{1}{w^2}$$

$$\text{or } w^2 = \frac{1}{w}$$

$$w^7 = w^6 \cdot w = (w^3)^2 \cdot w = 1^2 \cdot w = w$$

$$w^{29} = w^{27} \cdot w^2 = (w^3)^9 \cdot w^2 = 1^9 \cdot w^2 = w^2$$

$$w^{42} = (w^3)^{14} = 1^{14} = 1$$

Q. If $1, \omega, \omega^2$ are cube roots of unity then prove that

① $(1 + \omega^2)^4 = \omega$, ② $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$

③ $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$

④ $(2 + 5\omega + 2\omega^2)^6 = (2 + 2\omega + 5\omega^2)^6 = 729$

⑤ $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

Proof

① L.H.S. = $(1 + \omega^2)^4$
 $= (-\omega)^4$
 $= \omega^4 = \omega^3 \cdot \omega$
 $= 1 \cdot \omega = \omega$

② R.H.S. = $(1 - \omega + \omega^2)(1 + \omega - \omega^2)$
 $= (-\omega - \omega)(-\omega^2 - \omega^2)$
 $= (-2\omega)(-2\omega^2)$
 $= 4\omega^3 = 4 \cdot 1 = 4$

③ L.H.S. = $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5)$
 $= (1 - \omega)(1 - \omega^2)(1 - \omega^3 \cdot \omega)(1 - \omega^3 \cdot \omega^2)$
 $= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$
 $= \{(1 - \omega)(1 - \omega^2)\}^2 = (1 - \omega - \omega^2 + \omega^3)^2$
 $= \{1 - (\omega + \omega^2) + 1\}^2 = \{1 - (-1) + 1\}^2$
 $= 3^2 = 9$

④ L.H.S. = $(2 + 5\omega + 2\omega^2)^6$
 $= \{2(1 + \omega^2) + 5\omega\}^6$
 $= \{2(-\omega) + 5\omega\}^6$
 $= (-2\omega + 5\omega)^6$
 $= (3\omega)^6 = 3^6 \cdot (\omega^3)^2$
 $= 729 \cdot 1^2 = 729$

R.H.S. = $(2 + 2\omega + 5\omega^2)^6$
 $= \{2(1 + \omega) + 5\omega^2\}^6$
 $= \{2(-\omega^2) + 5\omega^2\}^6$
 $= (-2\omega^2 + 5\omega^2)^6$
 $= (3\omega^2)^6 = 3^6 \cdot \omega^{12}$
 $= 729 \cdot (\omega^3)^4 = 729 \cdot 1^4$
 $= 729$

⑤ $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$
 $= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$
 $= (-2\omega)^5 + (-2\omega^2)^5$
 $= -32\omega^5 + (-32\omega^{10})$
 $= -32[\omega^5 + \omega^{10}]$

R.H.S. = $-32(\omega^3 \cdot \omega^2 + \omega^9 \cdot \omega)$
 $= -32\{1 \cdot \omega^2 + (\omega^3)^3 \cdot \omega\}$
 $= -32(\omega^2 + 1 \cdot \omega)$
 $= -32(\omega^2 + \omega)$
 $= (-32)(-1) = 32$

De - Moivre's Theorem

If n is an integer, positive or negative or zero then $(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$

Proof Case I When n is a positive integer

$$\begin{aligned} \text{We have, } & (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= \cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 \\ &\quad + i^2 \sin \theta_1 \sin \theta_2 \\ &= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &\quad + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \end{aligned}$$

$$\begin{aligned} \text{Similarly, } & (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)(\cos \theta_3 + i \sin \theta_3) \\ &= \{ \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \} (\cos \theta_3 + i \sin \theta_3) \\ &= \cos (\theta_1 + \theta_2) \cdot \cos \theta_3 + i \sin (\theta_1 + \theta_2) \cdot \cos \theta_3 \\ &\quad + i \cos (\theta_1 + \theta_2) \cdot \sin \theta_3 + i^2 \sin (\theta_1 + \theta_2) \cdot \sin \theta_3 \\ &= \cos (\theta_1 + \theta_2) \cdot \cos \theta_3 - \sin (\theta_1 + \theta_2) \cdot \sin \theta_3 \\ &\quad + i \{ \sin (\theta_1 + \theta_2) \cdot \cos \theta_3 + \cos (\theta_1 + \theta_2) \cdot \sin \theta_3 \} \\ &= \cos (\theta_1 + \theta_2 + \theta_3) + i \sin (\theta_1 + \theta_2 + \theta_3) \end{aligned}$$

Proceeding in this way for n factors, we have,

$$\begin{aligned} & (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots (\cos \theta_n + i \sin \theta_n) \\ &= \cos (\theta_1 + \theta_2 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n) \quad \text{--- (1)} \end{aligned}$$

Putting $\theta_1 = \theta_2 = \dots = \theta$ in (1), we get

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Case - II When n is a negative integer

Let $n = -m$, where m is a positive integer

$$\begin{aligned} \text{Then } & (\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} \\ &= \frac{1}{(\cos \theta + i \sin \theta)^m} = \frac{1}{\cos m\theta + i \sin m\theta} \\ &= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta - i^2 \sin^2 m\theta} = \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \frac{\cos (-m\theta) + i \sin (-m\theta)}{1} = \cos n\theta + i \sin n\theta \end{aligned}$$

Case III When $n=0$, then $(\cos \theta + i \sin \theta)^0 = 1$

$$= \cos(0 \cdot \theta) + i \sin(0 \cdot \theta)$$

Thus, De Moivre's theorem is true for all integral value of n .

Remark The De-Moivre's theorem may also be used in the following manner when n is form

fraction ① $(\cos \theta + i \sin \theta)^{-n}$
 $= \cos(-n\theta) + i \sin(-n\theta)$
 $= \cos n\theta - i \sin n\theta$

Note
 1) $(\sin \theta \pm i \cos \theta)^n \neq \sin n\theta \pm i \cos n\theta$

② $(\cos \theta - i \sin \theta)^n = \{\cos(-\theta) + i \sin(-\theta)\}^n$
 $= \cos(n\theta) + i \sin(n\theta)$
 $= \cos n\theta + i \sin n\theta$

2) $(\cos \theta + i \sin \theta)^n \neq \cos n\theta + i \sin n\theta$

③ $(\cos \theta - i \sin \theta)^{-n} = \{\cos(-\theta) + i \sin(-\theta)\}^{-n}$
 $= \cos n\theta + i \sin n\theta$

④ $\frac{1}{\cos \theta + i \sin \theta} = (\cos \theta + i \sin \theta)^{-1}$
 $= \cos(-\theta) + i \sin(-\theta)$
 $= \cos \theta - i \sin \theta$

Q. simplify $\frac{(\cos 4\theta + i \sin 4\theta)^3 (\cos \theta - i \sin \theta)^{-4}}{(\cos \theta + i \sin \theta)^{-5} (\cos 2\theta + i \sin 2\theta)^{-6}}$
 $= \frac{(\cos 12\theta + i \sin 12\theta) \{\cos(-\theta) + i \sin(-\theta)\}^{-4}}{(\cos \theta + i \sin \theta)^{-5} \{(\cos \theta + i \sin \theta)^2\}^{-6}}$
 $= \frac{(\cos \theta + i \sin \theta)^{12} \{(\cos \theta + i \sin \theta)^{-1}\}^{-4}}{(\cos \theta + i \sin \theta)^{-5} (\cos \theta + i \sin \theta)^{-12}}$
 $= (\cos \theta + i \sin \theta)^{12+4+5+12} = (\cos \theta + i \sin \theta)^{33}$
 $= \cos 33\theta + i \sin 33\theta$

Q. simplify $\frac{(\cos \theta + i \sin \theta)^5}{(\sin \theta + i \cos \theta)^7} = (\cos \theta + i \sin \theta)^5 (\sin \theta + i \cos \theta)^{-7}$
 $= (\cos \theta + i \sin \theta)^5 \left\{ \cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right) \right\}^{-7} = \frac{(\cos 5\theta + i \sin 5\theta)^{-7}}{\cos\left(\frac{\pi}{2} - 7\theta\right) + i \sin\left(\frac{\pi}{2} - 7\theta\right)}$
 $= \cos\left(5\theta - \frac{7\pi}{2} + 7\theta\right) + i \sin\left(5\theta - \frac{7\pi}{2} + 7\theta\right)$
 $= \cos\left(\frac{7\pi}{2} - 12\theta\right) - i \sin\left(\frac{7\pi}{2} - 12\theta\right) = (-1)^{\frac{7+1}{2}} \sin(-12\theta) - i(-1)^{\frac{7-1}{2}} \cos(-12\theta)$
 $= -\sin 12\theta - i(-1) \cos 12\theta = -\sin 12\theta + i \cos 12\theta$

9) Q. If $x + \frac{1}{x} = 2 \cos \theta$ then show that

a) $x^n + \frac{1}{x^n} = 2 \cos n\theta$ b) $x^n - \frac{1}{x^n} = 2i \sin n\theta$

Proof

$$\begin{aligned}
 x + \frac{1}{x} &= 2 \cos \theta \\
 \Rightarrow x^2 + 1 &= 2x \cos \theta \\
 \Rightarrow x^2 - 2x \cos \theta + 1 &= 0 \\
 \Rightarrow x &= \frac{-(-2 \cos \theta) \pm \sqrt{(-2 \cos \theta)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
 &= \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} \\
 &= \frac{2 \cos \theta \pm \sqrt{-4(1 - \cos^2 \theta)}}{2} \\
 &= \frac{2 \cos \theta \pm 2 \sqrt{-\sin^2 \theta}}{2} \\
 &= \cos \theta \pm i \sin \theta
 \end{aligned}$$

Now, Taking

$$\begin{aligned}
 x &= \cos \theta + i \sin \theta \\
 x^n &= (\cos \theta + i \sin \theta)^n \\
 &= \cos n\theta + i \sin n\theta \\
 \frac{1}{x^n} &= x^{-n} = (\cos \theta + i \sin \theta)^{-n} \\
 &= \cos(-n\theta) + i \sin(-n\theta) \\
 &= \cos n\theta - i \sin n\theta
 \end{aligned}$$

$$\begin{aligned}
 \text{a) } x^n + \frac{1}{x^n} &= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta \\
 &= 2 \cos n\theta \\
 \text{b) } x^n - \frac{1}{x^n} &= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta \\
 &= 2i \sin n\theta
 \end{aligned}$$

Again, taking

$$\begin{aligned}
 x &= \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta) \\
 x^n &= \{\cos(-\theta) + i \sin(-\theta)\}^n \\
 &= \cos(-n\theta) + i \sin(-n\theta) \\
 &= \cos n\theta - i \sin n\theta \\
 \frac{1}{x^n} &= x^{-n} = \{\cos(-\theta) + i \sin(-\theta)\}^{-n} \\
 &= \cos n\theta + i \sin n\theta
 \end{aligned}$$

$$\text{a) } x^n + \frac{1}{x^n} = 2 \cos n\theta \quad \text{b) } x^n - \frac{1}{x^n} = -2i \sin n\theta$$

Q. Given $\cos X + \cos B + \cos Y = \sin X + \sin B + \sin Y = 0$

a) Prove that $\cos 3X + \cos 3B + \cos 3Y = 3 \cos (X+B+Y)$

$\sin 3X + \sin 3B + \sin 3Y = 3 \sin (X+B+Y)$

b) Prove that $\cos^2 X + \cos^2 B + \cos^2 Y = \sin^2 X + \sin^2 B + \sin^2 Y = \frac{3}{2}$

Proof Let $x = \cos X + i \sin X$, $y = \cos B + i \sin B$, $z = \cos Y + i \sin Y$

$$x + y + z = (\cos X + \cos B + \cos Y) + i(\sin X + \sin B + \sin Y)$$

$$= 0 + i \cdot 0 = 0$$

a) We know, $x^3 + y^3 + z^3 - 3xyz = 0$

$$\Rightarrow x^3 + y^3 + z^3 = 3xyz$$

$$\Rightarrow (\cos X + i \sin X)^3 + (\cos B + i \sin B)^3 + (\cos Y + i \sin Y)^3$$

$$= 3(\cos X + i \sin X)(\cos B + i \sin B)(\cos Y + i \sin Y)$$

$$\Rightarrow (\cos 3X + i \sin 3X) + (\cos 3B + i \sin 3B) + (\cos 3Y + i \sin 3Y)$$

$$= 3 \{ \cos(X+B+Y) + i \sin(X+B+Y) \}$$

$$\Rightarrow (\cos 3X + \cos 3B + \cos 3Y) + i(\sin 3X + \sin 3B + \sin 3Y)$$

$$= 3 \{ \cos(X+B+Y) + i \sin(X+B+Y) \}$$

$$\therefore \cos 3X + \cos 3B + \cos 3Y = 3 \cos(X+B+Y)$$

$$\text{and } \sin 3X + \sin 3B + \sin 3Y = 3 \sin(X+B+Y)$$

b) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = x^{-1} + y^{-1} + z^{-1}$

$$\Rightarrow \frac{yz + zx + xy}{xyz} = (\cos X + i \sin X)^{-1} + (\cos B + i \sin B)^{-1} + (\cos Y + i \sin Y)^{-1}$$

$$= (\cos X - i \sin X) + (\cos B - i \sin B) + (\cos Y - i \sin Y)$$

$$= (\cos X + \cos B + \cos Y) - i(\sin X + \sin B + \sin Y)$$

$$= 0 - i \cdot 0 = 0$$

$$\Rightarrow yz + zx + xy = 0$$

$$\text{Now, } (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$\Rightarrow 0^2 = x^2 + y^2 + z^2$$

$$\Rightarrow 0 = (\cos X + i \sin X)^2 + (\cos B + i \sin B)^2 + (\cos Y + i \sin Y)^2$$

$$\Rightarrow 0 = (\cos 2X + i \sin 2X) + (\cos 2B + i \sin 2B) + (\cos 2Y + i \sin 2Y)$$

$$\Rightarrow 0 + i \cdot 0 = (\cos 2X + \cos 2B + \cos 2Y) + i(\sin 2X + \sin 2B + \sin 2Y)$$

$$\therefore \cos 2X + \cos 2B + \cos 2Y = 0 \text{ and } \sin 2X + \sin 2B + \sin 2Y = 0$$

$$\Rightarrow 2\cos^2 X - 1 + 2\cos^2 B - 1 + 2\cos^2 Y - 1 = 0 \Rightarrow 1 - 2\sin^2 X + 1 - 2\sin^2 B + 1 - 2\sin^2 Y = 0$$

$$\Rightarrow 2(\cos^2 X + \cos^2 B + \cos^2 Y) = 3 \Rightarrow \sin^2 X + \sin^2 B + \sin^2 Y = \frac{3}{2}$$

Rank of a matrix

- A matrix is said to be of rank r when
- (i) it has at least one non-zero minor of order r and
 - (ii) every minor of order higher than r vanishes.

Elementary transformation of a matrix

- (i) The interchange of any two rows (columns)
- (ii) The multiplication of any row (column) by a non-zero number.
- (iii) The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

These three rows (columns) are known as elementary transformations of a matrix.

{ Elementary transformations do not change either the order or rank of a matrix.

Equivalent matrix

Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.

Two equivalent matrices have the same order and the same rank. The symbol \sim is used for equivalence.

Q. Determine the rank of

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solⁿ

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Replacing R_1 and R_3
by $R_1 - R_2$ and $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Replacing C_2 and C_3 by
 $\frac{1}{2}C_2$ and $C_3 - 4C_1 + \frac{1}{2}C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The 1st order minor is zero
The 2nd " " is not zero,

$$\text{Let } \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

Hence the rank of the matrix A is 2.

Q Determine the rank of

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solⁿ Let A =

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Replacing C_3 and C_4
by $C_3 - C_1$ and $C_4 - C_1$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$$

Replacing R_2 and R_4
by $R_2 - R_1$ and $R_4 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Replacing c_3 and c_4 by $c_3 + 3c_2$ and $c_4 + c_2$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

\therefore The 4th order minor is zero.
 Also, every 3rd " " is zero.
 But, 2nd " " is not zero,

ie, $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$

Hence the rank of the matrix A is 2.

Consistency of Linear System of equations

Let the system of m linear equations

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= k_2 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= k_m \end{aligned} \right\}$$

Containing the n unknowns x_1, x_2, \dots, x_n .

Here $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

is called the Coefficient matrix.

and $k = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$ is called the augmented matrix.

Test the consistency of a system of equations in n unknowns

Find the ranks of the coefficient matrix A and the augmented matrix k , by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of k be r' .

- (i) If $r \neq r'$, the equations are inconsistent, i.e., there is no solution.
- (ii) If $r = r' = n$, the equations are consistent and there is a unique solution.
- (iii) If $r = r' < n$, the equations are consistent and there are infinite number of solutions.

Q. Test for consistency and solve

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

Sol.ⁿ

We have,

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Replacing R_1 and R_2 by $3R_1$ and $5R_2$

$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$$

Replacing R_2 by $R_2 - R_1$

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$$

Replacing R_1, R_2 and R_3 by $\frac{1}{3}R_1, \frac{1}{11}R_2$ and $5R_3$

$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$$

Replacing R_1 and R_3 by $\frac{1}{7}R_1$ and $R_3 + R_2 - R_1$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

\therefore The rank of coefficient matrix = The rank of augmented matrix = 2

Hence the equations are consistent.

Also, $5x + 3y + 7z = 4$ ——— ①

and $11y - z = 3$

$\Rightarrow 11y = 3 + z$

$\Rightarrow y = \frac{3}{11} + \frac{z}{11}$ ——— ②

From ①, $5x + \frac{9}{11} + \frac{3z}{11} + 7z = 4$

$\Rightarrow 5x = 4 - \frac{9}{11} - 7z - \frac{3z}{11}$

$= \frac{35}{11} - \frac{80}{11}z$

$\Rightarrow x = \frac{7}{11} - \frac{16}{11}z$

where z is a parameter.

Hence $x = \frac{7}{11}, y = \frac{3}{11}$ and $z = 0$

is a particular solution.

Q. Investigate the values of λ and μ so that the equations

$$2x + 3y + 5z = 9$$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

have (i) a unique solution, (ii) no solution and (iii) an infinite number of solutions.

Sol. We have,

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Replacing R_2 and R_3 by $R_2 - 3R_1$, $R_3 - R_1$,

$$\begin{bmatrix} 2 & 3 & 5 \\ 1 & -6 & -17 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -19 \\ \mu - 9 \end{bmatrix}$$

Replacing R_{12}

$$\begin{bmatrix} 1 & -6 & -17 \\ 2 & 3 & 5 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -19 \\ 9 \\ \mu - 9 \end{bmatrix}$$

Replacing R_2 by $R_2 - 2R_1$

$$\begin{bmatrix} 1 & -6 & -17 \\ 0 & 15 & 39 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -19 \\ 47 \\ \mu - 9 \end{bmatrix}$$

Let the rank of the coefficient matrix = $\rho(A)$
and the rank of the augmented matrix = $\rho(k)$

(i) Unique Solution

The given system will have a unique solution, if $\rho(A) = \rho(k) = 3$

This is only possible if $\lambda - 5 \neq 0$, μ may have any value.

Hence the unique solution $\lambda \neq 5$ and μ may have any value.

(ii) No solution

The given system will have no solution if $f(A) \neq f(k)$, i.e., $f(A) = 2$ and $f(k) = 3$. This is only possible if $\lambda - 5 = 0$ and $\mu - 9 \neq 0$. Hence there is no solution $\lambda = 5$ and $\mu \neq 9$.

(iii) Infinite number of solutions

The given system will have an infinite number of solutions if $f(A) = f(k) = 2 < \text{no. of unknowns}$.

This is only possible if $\lambda - 5 = 0$ and $\mu - 9 = 0$, i.e., $\lambda = 5$, $\mu = 9$.

Hence there are infinite numbers of solutions $\lambda = 5$ and $\mu = 9$.

Linear Differential Equations

Def.ⁿ Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = X$$

Where $P_1, P_2, \dots, P_{n-1}, P_n$ and X are functions of x only.

Def.ⁿ Linear differential equations with constant coefficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

Where $k_1, k_2, \dots, k_{n-1}, k_n$ are constants and X are ~~some~~ functions of x only.

Working Procedure to solve the equation

Symbolic form is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_{n-1} D + k_n) y = X$$

$$\Rightarrow [f(D)] y = X$$

Auxiliary equation is

$$f(D) = 0 \text{ and solve it for } D.$$

Write the complementary function (C.F.) as follows:—

Roots of A.E.	C.F
1. m_1, m_2, m_3, \dots	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots$
2. m_1, m_1, m_2, \dots	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_2 x} + \dots$
3. $m_1, m_1, m_1, m_2, \dots$	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_2 x} + \dots$
4. $\alpha \pm i\beta, m_1, m_2, \dots$	$e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_1 x} + c_4 e^{m_2 x} + \dots$

$$5. \kappa \pm i\beta, \kappa \pm i\beta, m_1, m_2, \dots$$

$$e^{\kappa x} \left[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x \right] + c_5 e^{m_1 x} + c_6 e^{m_2 x} + \dots$$

$$6. \kappa \pm i\beta, a \pm ib, m_1, m_2, \dots$$

$$e^{\kappa x} (c_1 \cos \beta x + c_2 \sin \beta x) + e^{ax} (c_3 \cos bx + c_4 \sin bx) + c_5 e^{m_1 x} + c_6 e^{m_2 x} + \dots$$

Q. Solve $\frac{d^2y}{dx^2} - k^2y = 0$

Sol.ⁿ Symbolic form is

$$D^2y - k^2y = 0$$

$$\Rightarrow (D^2 - k^2)y = 0$$

Auxiliary equation is

$$D^2 - k^2 = 0$$

$$\Rightarrow D^2 = k^2$$

$$\Rightarrow D = \pm k$$

$$\text{C.F.} = c_1 e^{kx} + c_2 e^{-kx}$$

Hence the complete solution

$$\text{is } y = c_1 e^{kx} + c_2 e^{-kx}$$

Q. Solve $\frac{d^3y}{dx^3} + y = 0$

Sol.ⁿ Symbolic form is

$$D^3y + y = 0$$

$$\Rightarrow (D^3 + 1)y = 0$$

Auxiliary equation is

$$D^3 + 1 = 0$$

$$\Rightarrow (D+1)(D^2 - D + 1) = 0$$

$$\therefore D+1=0 \text{ or } D^2-D+1=0$$

$$\Rightarrow D = -1$$

$$D = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) e^{\frac{1}{2}x} + c_3 e^{-x}$$

Hence the complete solution

$$\text{is } y = \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) e^{x/2} + c_3 e^{-x}$$

Q. Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Sol. Symbolic form is

$$D^2y + 4Dy + 13y = 0$$

$$\Rightarrow (D^2 + 4D + 13)y = 0$$

Auxiliary equation is

$$D^2 + 4D + 13 = 0$$

$$\begin{aligned} \therefore D &= \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} & \left. \begin{aligned} \sqrt{-1} &= i \\ -1 &= i^2 \\ \sqrt{-36} &= \sqrt{-1} \sqrt{36} \\ &= i \cdot 6 \end{aligned} \right\} \\ &= \frac{-4 \pm \sqrt{16 - 52}}{2} & \\ &= \frac{-4 \pm \sqrt{-36}}{2} & \\ &= \frac{-4 \pm 6i}{2} & \left. \begin{aligned} \alpha \pm i\beta \\ \alpha &= -2 \\ \beta &= 3 \end{aligned} \right\} \\ &= -2 \pm 3i & \end{aligned}$$

$$C.F. = (c_1 \cos 3x + c_2 \sin 3x) e^{-2x}$$

Hence the complete solution is $y = (c_1 \cos 3x + c_2 \sin 3x) e^{-2x}$

To find the Particular Integral (P.I.)

$$P.I. = \left\{ \frac{1}{f(D)} \text{ or } \frac{1}{f(D^2)} \right\} X$$

① When $X = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax}, \text{ Put } D=a, f(a) \neq 0$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ Put } D=a, f(a)=0, f'(a) \neq 0$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ Put } D=a, f'(a)=0, f''(a) \neq 0$$

and so on.

Q. Find the P.I. of $(D^2+5D+6)y = e^x$

$$\left[\begin{aligned} \sinh ax &= \frac{e^{ax} - e^{-ax}}{2} \\ \cosh ax &= \frac{e^{ax} + e^{-ax}}{2} \end{aligned} \right.$$

$$\checkmark b) (D+2)y = e^{-2x}$$

$$\checkmark c) (D-1)^2 y = 2 \sinh x$$

$$\checkmark d) (D^2+16)y = e^{-4x}$$

$$e) (D^2-4)y = e^{2x} + e^{-4x}$$

$$\checkmark f) (D^3+3D^2+3D+1)y = e^{-x}$$

② When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I. = \frac{1}{f(D^2)} \{ \sin(ax+b) \text{ or } \cos(ax+b) \} \text{ Put } D^2 = -a^2, f(-a^2) \neq 0$$

$$= x \frac{1}{f'(D^2)} \{ \text{"} \} \text{ Put } D^2 = -a^2, f(-a^2) = 0, f'(-a^2) \neq 0$$

$$= x^2 \frac{1}{f''(D^2)} \{ \text{"} \} \text{ Put } D^2 = -a^2, f'(-a^2) = 0, f''(-a^2) \neq 0$$

and so on.

Q. Find the P.I. of $(D^2+D+1)y = \sin 2x$

$$\checkmark b) (D^2+9)y = 4 \sin 3x$$

$$c) (D^2+9)y = \cos 3x$$

$$\checkmark d) (D^3+1)y = \cos 2x$$

$$\checkmark e) (D^2+4D)y = \sin 2x$$

③ When $x = e^{ax} V$, where V is a function of x

$$P.I. = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$ as in ① and ②.

Q. Find the P.I. of a) $(D^2 - 2D + 4) y = e^x \cos x$

b) $(D^2 - 2D + 2) y = e^x \sin x$

④ When x is any function of x

P.I. = $\frac{1}{f(D)} x$ Resolve $\frac{1}{f(D)}$ into partial fractions

and operate each partial fraction on x

remembering that $\frac{1}{D-a} x = e^{ax} \int x e^{-ax} dx$

If $x \neq 0$, then the complete solution is $y = C.F. + P.I.$

To find the Particular Integral (P.I.)

$$P.I. = \left\{ \frac{1}{f(D)} \text{ or } \frac{1}{f(D^2)} \right\} X$$

① When $x = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} \quad \text{Put } D = a, f(a) \neq 0$$

$$= x \frac{1}{f'(D)} e^{ax} \quad \text{Put } D = a, f(a) = 0, f'(a) \neq 0$$

$$= x^2 \frac{1}{f''(D)} e^{ax} \quad \text{Put } D = a, f'(a) = 0, f''(a) \neq 0$$

and so on.

Q. Find the Particular Integral of

$$(D^2 + 5D + 6) y = e^x$$

Sol. P.I. = $\frac{1}{D^2 + 5D + 6} e^x$

$$= \frac{1}{1^2 + 5 \cdot 1 + 6} e^x$$

$$= \frac{1}{1 + 5 + 6} e^x$$

$$= \frac{1}{12} e^x$$

Rough
 $e^{ax} = e^x$
 $a = 1$

put $D = a = 1$

Q. Find the Particular Integral of
 $(D^2 + 16)y = e^{-4x}$

Sol. P.I. = $\frac{1}{D^2 + 16} e^{-4x}$ Put $D = a = -4$

$$= \frac{1}{(-4)^2 + 16} e^{-4x}$$

$$= \frac{1}{16 + 16} e^{-4x}$$

$$= \frac{1}{32} e^{-4x}$$

Note
 $\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

Q. Find the P.I. of
 $(D + 2)y = e^{-2x}$

Sol. P.I. = $\frac{1}{D + 2} e^{-2x}$ Put $D = a = -2$

$$= x \frac{1}{1 + 0} e^{-2x}$$

$$= x e^{-2x}$$

Q. Find the P.I. of

$$(D - 1)^2 y = 2 \sinh x$$

Sol.

~~$(D - 1)^2 y = 2 \sinh x$~~
P.I. = $\frac{1}{(D - 1)^2} \cdot 2 \sinh x$

$$= \frac{1}{(D - 1)^2} \cdot 2 \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{(D - 1)^2} e^x - \frac{1}{(D - 1)^2} e^{-x}$$

$$\text{Put } D = a = 1$$

$$D = a = -1$$

$$= x \frac{1}{2(D-1) \cdot 1} e^x - \frac{1}{(-1-1)^2} e^{-x}$$

$$= x^2 \frac{1}{2 \cdot 1} e^x - \frac{1}{4} e^{-x}$$

$$= \frac{1}{2} x^2 e^x - \frac{1}{4} e^{-x}$$

Q. Find the P.I. of $(D+1)^3 y = e^{-x}$

Sol. P.I. = $\frac{1}{(D+1)^3} e^{-x}$ Put $D = a = -1$

$$= x \frac{1}{3(D+1)^2 \cdot 1} e^{-x}$$

$$= x^2 \frac{1}{3 \cdot 2(D+1) \cdot 1} e^{-x}$$

$$= x^3 \frac{1}{6 \cdot 1} e^{-x}$$

$$= \frac{1}{6} x^3 e^{-x}$$

(2) When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$\text{P.I.} = \frac{1}{f(D^2)} \{ \sin(ax+b) \text{ or } \cos(ax+b) \}$$

$$\text{Put } D^2 = -a^2, f(-a^2) \neq 0$$

$$= x \frac{1}{f'(D^2)} \{ \sin(ax+b) \text{ or } \cos(ax+b) \}$$

$$\text{Put } D^2 = -a^2, f(-a^2) = 0,$$

$$f'(-a^2) \neq 0$$

$$= x^2 \frac{1}{f''(D^2)} \{ \sin(ax+b) \text{ or } \cos(ax+b) \}$$

$$\text{Put } D^2 = -a^2, f'(-a^2) = 0,$$

$$f''(-a^2) \neq 0$$

and so on.

Q. find the P.I. of $(D^2 + D + 1)y = \sin 2x$

Sol.ⁿ

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \sin 2x \quad \text{Put } D^2 = -a^2$$

$$= \frac{1}{-4 + D + 1} \sin 2x \quad = -2^2$$

$$= \frac{1}{D - 3} \sin 2x \quad = -4$$

$$= \frac{D + 3}{D^2 - 9} \sin 2x$$

$$= \frac{D + 3}{-4 - 9} \sin 2x$$

$$= -\frac{1}{13} \{ D(\sin 2x) + 3 \sin 2x \}$$

$$= -\frac{1}{13} \{ \cos 2x \cdot 2 + 3 \sin 2x \}$$

$$= -\frac{1}{13} (2 \cos 2x + 3 \sin 2x)$$

③ When $X = e^{ax} V$, where V is a function of x

$$P.I. = \frac{1}{f(D)} e^{ax} V \quad \text{Put } D = D+a$$

$$= e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)} V$

as in ① and ②.

Q. Find the Particular Integral of
 $(D^2 - 2D + 4)y = e^x \cos x$

Sol. P.I. = $\frac{1}{D^2 - 2D + 4} e^x \cos x$
Put $D = D+a$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x \quad \text{Put } D^2 = -1^2 = -1$$

$$= e^x \frac{1}{-1 + 3} \cos x$$

$$= \frac{1}{2} e^x \cos x$$

Q. Find the Particular Integral of

$$(D^2 - 2D + 2)y = e^x \sin x$$

Sol.ⁿ

$$P.I. = \frac{1}{D^2 - 2D + 2} \cdot e^x \sin x \quad \text{put } D = D+1$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \sin x$$

$$= e^x \frac{1}{D^2 + \cancel{2D} + 1 - \cancel{2D} - \cancel{2} + \cancel{2}} \sin x$$

$$= e^x \frac{1}{D^2 + 1} \sin x \quad \begin{aligned} D^2 &= -1^2 \\ &= -1 \end{aligned}$$

$$= e^x x \frac{1}{2D} \sin x$$

$$= \frac{x e^x}{2} \int \sin x \, dx$$

$$= \frac{1}{2} x e^x (-\cos x)$$

$$= -\frac{1}{2} x e^x \cos x$$

Q. Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x)$

Sol.ⁿ Auxiliary equation is

$$(D-2)^2 = 0$$

$$\Rightarrow D-2 = 0 \quad (\text{two times})$$

$$\therefore D = 2, 2$$

$$\text{C.F.} = (c_1 + c_2 x) e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} 8(e^{2x} + \sin 2x)$$

$$= 8 \left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x \right]$$

Put $D = 2$ Put $D^2 = -2^2 = -4$

$$= 8 \left[x \frac{1}{2(D-2) \cdot 1} e^{2x} + \frac{1}{D^2 - 4D + 4} \sin 2x \right]$$

$$= 8 \left[x^2 \frac{1}{2 \cdot 1} e^{2x} + \frac{1}{-4 - 4D + 4} \sin 2x \right]$$

$$= 8 \left[\frac{1}{2} x^2 e^{2x} + \frac{1}{-4D} \sin 2x \right]$$

$$= 4x^2 e^{2x} - 2 \int \sin 2x \, dx$$

$$= 4x^2 e^{2x} - 2 \frac{-\cos 2x}{2}$$

$$= 4x^2 e^{2x} + \cos 2x$$

Hence the complete solution is

$$y = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x$$

Q. Solve $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = x e^x \sin x$

Sol.

Symbolic form is

$$D^2 y - 2Dy + y = x e^x \sin x$$

$$\Rightarrow (D^2 - 2D + 1)y = e^x (x \sin x)$$

$$\Rightarrow (D-1)^2 y = e^x (x \sin x)$$

Auxiliary equation is

$$(D-1)^2 = 0$$

$$\Rightarrow D-1 = 0 \quad (\text{two times})$$

$$\therefore D = 1, 1$$

$$e.f. = (c_1 + c_2 x) e^x$$

$$P.I. = \frac{1}{(D-1)^2} e^x (x \sin x)$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D} \int x \sin x dx$$

$$= e^x \frac{1}{D} \left[x \int \sin x dx - \int (x)' \left\{ \int \sin x dx \right\} dx \right]$$

$$= e^x \frac{1}{D} \left[x (-\cos x) - \int 1 \cdot (-\cos x) dx \right]$$

$$= e^x \frac{1}{D} \left[-x \cos x + \sin x \right]$$

$$= e^x \int (-x \cos x + \sin x) dx$$

$$\begin{aligned}
&= e^x \left[- \left(x \int \cos x \, dx - \int (x)' \int \cos x \, dx \right) + (-\cos x) \right] \\
&= e^x \left[- \left(x \sin x - \int 1 \cdot \sin x \, dx \right) - \cos x \right] \\
&= e^x \left[-x \sin x + (-\cos x) - \cos x \right] \\
&= e^x \left[-x \sin x - 2 \cos x \right] \\
&= -e^x (x \sin x + 2 \cos x)
\end{aligned}$$

Hence the complete solution is

$$y = (c_1 + c_2 x) e^x - e^x (x \sin x + 2 \cos x)$$

Q. solve $\frac{d^3 y}{dx^3} + 2 \frac{d^2 y}{dx^2} + \frac{dy}{dx} = e^{2x} + \cos 2x$

Sol. Symbolic form is

$$\begin{aligned}
D^3 y + 2D^2 y + Dy &= e^{2x} + \cos 2x \\
\Rightarrow (D^3 + 2D^2 + D) y &= e^{2x} + \cos 2x
\end{aligned}$$

Auxiliary equation is

$$\begin{aligned}
D^3 + 2D^2 + D &= 0 \\
\Rightarrow D(D^2 + 2D + 1) &= 0
\end{aligned}$$

$$\Rightarrow D(D+1)^2 = 0$$

$$\therefore D = 0, -1, -1$$

$$C.F. = c_1 e^{0 \cdot x} + (c_2 + c_3 x) e^{-x}$$

$$= C_1 + (C_2 + C_3 x) e^{-x}$$

$$P.I. = \frac{1}{D^3 + 2D^2 + D} (e^{2x} + \sin 2x)$$

$$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

$$\text{Put } D = 2$$

$$\text{Put } D^2 = -2^2 = -4$$

$$= \frac{1}{2^3 + 2 \cdot 2^2 + 2} e^{2x} + \frac{1}{D^2 \cdot D + 2(-4) + D} \sin 2x$$

$$= \frac{1}{18} e^{2x} + \frac{1}{-4D - 8 + D} \sin 2x$$

$$= \frac{1}{18} e^{2x} - \frac{1}{3D + 8} \sin 2x$$

$$= \frac{1}{18} e^{2x} - \frac{3D - 8}{9D^2 - 64} \sin 2x$$

$$= \frac{1}{18} e^{2x} - \frac{3D - 8}{9(-4) - 64} \sin 2x$$

$$= \frac{1}{18} e^{2x} - \frac{1}{-100} \{3D(\sin 2x) - 8 \sin 2x\}$$

$$= \frac{1}{18} e^{2x} + \frac{1}{100} (3 \cos 2x \cdot 2 - 8 \sin 2x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{50} (3 \cos 2x - 4 \sin 2x)$$

Hence the complete solution is

$$y = C_1 + (C_2 + C_3 x) e^{-x} + \frac{1}{18} e^{2x}$$

$$+ \frac{1}{50} (3 \cos 2x - 4 \sin 2x)$$

Partial Differential Equations

Equations involving ^{one or more} partial derivatives are called partial differential equations.

$z = f(x, y)$, x and y will be taken as the independent variables and z , the dependent variable.

Notation $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

Formation of Partial Differential Equations

The partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.

Q. Derive a partial differential equation (by eliminating the constants) from the equation

$$(i) \quad z = ax + by + a^2 + b^2$$

Solⁿ: $z = ax + by + a^2 + b^2$ ——— (1)

Differentiating (1) partially w.r.t. x and y ,

$$\frac{\partial z}{\partial x} = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b$$

$$\Rightarrow p = a$$

$$\Rightarrow q = b$$

Putting the values of a and b in (1),

$$z = px + qy + p^2 + q^2$$

is the required partial differential equation.

$$(ii) \quad 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$(iii) \quad (x-a)^2 + (y-b)^2 + z^2 = c^2$$

$$(iv) \quad z = (x^2 + a)(y^2 + b)$$

Q. Form the partial differential equations (by eliminating the arbitrary functions) from the eqnⁿ,

a) $z = f(x^2 - y^2)$

b) $z = f\left(\frac{xy}{z}\right)$

c) $z = f(x+at) + g(x-at)$

Sol.ⁿ a) $z = f(x^2 - y^2)$ ——— ①

Differentiating ① partially w.r.t. x and y ,

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2) \cdot 2x \quad \text{and} \quad q = \frac{\partial z}{\partial y} = f'(x^2 - y^2) \cdot (-2y) \quad \text{--- ②}$$

Dividing ② by ③, $\frac{p}{q} = \frac{f'(x^2 - y^2) \cdot 2x}{f'(x^2 - y^2) \cdot (-2y)} = -\frac{x}{y}$

$$\Rightarrow py = -qx$$

$$\Rightarrow py + qx = 0$$

is the required partial differential equation.

Sol.ⁿ c) $z = f(x+at) + g(x-at)$ ——— ①

Differentiating ① partially w.r.t. x and t ,

$$\frac{\partial z}{\partial x} = f'(x+at) + g'(x-at)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + g''(x-at)$$

and $\frac{\partial z}{\partial t} = f'(x+at) \cdot a + g'(x-at) \cdot (-a)$

$$= a \{ f'(x+at) - g'(x-at) \}$$

$$\frac{\partial^2 z}{\partial t^2} = a \{ f''(x+at) \cdot a - g''(x-at) \cdot (-a) \}$$

$$= a^2 \{ f''(x+at) + g''(x-at) \}$$

$$= a^2 \frac{\partial^2 z}{\partial x^2}$$

is the required partial differential equation.

Linear Equation of the first order

Def.ⁿ A linear partial differential equation of the first order, commonly known as Lagrange's Linear Equation is of the form $Pp + Qq = R$

Where P, Q and R are functions of x, y and z .

Method of solving Lagrange's Linear Equation

Equation of the form
 $Pp + Qq = R$

Ist step

Form the auxiliary (or subsidiary) equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

IInd step

Solve these auxiliary equations.

There are three methods for solving the above auxiliary equations.

First Method

Take two members and solve this equation, then take two other members and solve that equation.

Second Method

Using method of multipliers

$$\text{let } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{L dx + m dy + n dz}{LP + mQ + nR} \quad \text{--- (I)}$$

$$\text{and } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{L' dx + m' dy + n' dz}{L'P + m'Q + n'R} \quad \text{--- (II)}$$

Choose L, m, n and L', m', n' (not necessarily constants) are known as multipliers,

$$\text{So that } LP + mQ + nR = 0$$

$$\text{and } L'P + m'Q + n'R = 0$$

From (I) and (II), we have,

$$L dx + m dy + n dz = 0 \quad \text{--- (III)}$$

$$\text{and } L' dx + m' dy + n' dz = 0 \quad \text{--- (IV)}$$

Solve the equations (III) and (IV).

Third Method

Obtain one solution by applying first method and the other solution by applying second method.

IIIrd Step

Let ~~the~~ the two solutions be

$$u = c_1 \quad \text{and} \quad v = c_2$$

$$\text{Then} \quad f(u, v) = 0$$

$$\text{or} \quad u = f(v)$$

$$\text{or} \quad v = f(u)$$

is the required solution,

$$\textcircled{1} \quad \text{Solve} \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

Solⁿ ~~the~~ Here auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Now, taking the first two members,

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\text{Integrating,} \quad \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \log x = \log y + \log c_1$$

$$= \log y c_1$$

$$\Rightarrow x = y c_1$$

$$\Rightarrow \frac{x}{y} = c_1 \quad \text{--- (1)}$$

Again, taking last two members,

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating, $\int \frac{dy}{y} = \int \frac{dx}{x}$

$$\Rightarrow \log y = \log x + \log C_2$$
$$= \log x C_2$$

$$\Rightarrow y = x C_2$$

$$\Rightarrow \frac{y}{x} = C_2 \quad \text{--- (ii)}$$

Hence from (i) and (ii),

$$f\left(\frac{x}{y}, \frac{y}{x}\right) = 0$$

is the required solution.

(2) Solve $yzp + xxq = xy$

Sol. Here auxiliary equations are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Now, taking 1st and 2nd members,

$$\frac{dx}{yz} = \frac{dy}{zx}$$

$$\Rightarrow x dx = y dy$$

Integrating, $\int x dx = \int y dy$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + \frac{C_1}{2}$$

$$\Rightarrow x^2 = y^2 + c_1$$

$$\Rightarrow x^2 - y^2 = c_1 \quad \text{--- (I)}$$

Again, taking 2nd and 3rd members,

$$\frac{dy}{x^2} = \frac{dx}{xy}$$

$$\Rightarrow y dy = x dx$$

Integrating, $\int y dy = \int x dx$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{c_2}{2}$$

$$\Rightarrow y^2 = x^2 + c_2$$

$$\Rightarrow y^2 - x^2 = c_2 \quad \text{--- (II)}$$

Hence from (I) and (II),

$$x^2 - y^2 = f(y^2 - x^2)$$

is the required solution,

Q. Solve $x^2 y p + x^2 x q = y^2 x$

Solⁿ Here auxiliary equations are

$$\frac{dx}{x^2 y} = \frac{dy}{x^2 x} = \frac{dx}{y^2 x}$$

Now, taking 1st and 2nd members,

$$\frac{dx}{x^2 y} = \frac{dy}{x^2 x}$$

$$\Rightarrow x dx = y dy$$

Integrating, $\int x dx = \int y dy$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + \frac{C_1}{2}$$

$$\Rightarrow x^2 = y^2 + C_1$$

$$\Rightarrow x^2 - y^2 = C_1 \quad \text{--- (i)}$$

Again taking 2nd and 3rd members

$$\frac{dy}{z^2 x} = \frac{dz}{y^2 x}$$

$$\Rightarrow y^2 dy = z^2 dz$$

Integrating, $\int y^2 dy = \int z^2 dz$

$$\Rightarrow \frac{y^3}{3} = \frac{z^3}{3} + \frac{C_2}{3}$$

$$\Rightarrow y^3 = z^3 + C_2$$

$$\Rightarrow y^3 - z^3 = C_2 \quad \text{--- (ii)}$$

Hence from (i) and (ii),

$$y^3 - z^3 = f(x^2 - y^2)$$

is the required solution.

Q. Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = (x^2 - y^2)z$

Solⁿ Here auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{(x^2 - y^2)z}$$

Now, using multipliers x , y and z ,

$$\text{Each member} = \frac{x dx + y dy + z dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$$

$$= \frac{x dx + y dy + z dz}{x^2 y^2 - z^2 x^2 + y^2 z^2 - x^2 y^2 + z^2 x^2 - y^2 z^2}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$\Rightarrow 0 = x dx + y dy + z dz$$

Integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$

$$\Rightarrow x^2 + y^2 + z^2 = c_1 \quad \text{--- (1)}$$

Again, using multipliers $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$,

$$\therefore \text{Each members} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$= \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

$$\Rightarrow 0 = \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz$$

Integrating, $\log x + \log y + \log z = \log c_2$

$$\Rightarrow \log xyz = \log c_2$$

$$\Rightarrow xyz = c_2 \quad \text{--- (11)}$$

Hence from (1) and (11),

$$f(x^2 + y^2 + z^2, xyz) = 0$$

is the required solution.

Q. Solve $(y+z)p - (z+x)q = x-y$

Sol. Here auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{-(z+x)} = \frac{dz}{x-y}$$

Now, using multipliers 1, 1, 1

$$\therefore \text{Each members} = \frac{dx + dy + dz}{y+z - z-x + x-y}$$

$$= \frac{dx + dy + dz}{0}$$

$$\Rightarrow 0 = dx + dy + dz$$

Integrating, $x + y + z = c_1$ — (1)

Again, using multipliers $x, y, -z,$

$$\begin{aligned} \therefore \text{Each members} &= \frac{x dx + y dy - z dz}{x(y+z) - y(x+z) - z(x+y)} \\ &= \frac{x dx + y dy - z dz}{xy + zx - yx - xz - zx + yz} \\ &= \frac{x dx + y dy - z dz}{0} \end{aligned}$$

$$\Rightarrow 0 = x dx + y dy - z dz$$

$$\text{Integrating, } \frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = \frac{c_2}{2}$$

$$\Rightarrow x^2 + y^2 - z^2 = c_2 \text{ — (II)}$$

Hence from (I) and (II),

$$x + y + z = f(x^2 + y^2 - z^2)$$

is the required solution.

Gamma function
(or Euler's integral of
second kind)

The gamma function is defined
as $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, (n > 0)$

Reduction formula for Gamma
function

$$\Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx$$

$$= \left[x^n \int e^{-x} dx - \int (x^n)' \left\{ \int e^{-x} dx \right\} dx \right]_0^{\infty}$$

$$= \left[x^n \cdot \frac{e^{-x}}{-1} - \int n x^{n-1} \frac{e^{-x}}{-1} dx \right]_0^{\infty}$$

$$= - \left[\infty^n \cdot e^{-\infty} - 0^n \cdot e^{-0} - n \int_0^{\infty} x^{n-1} e^{-x} dx \right]$$

$$= - \left[0 - 0 - n \Gamma(n) \right]$$

$$= n \Gamma(n)$$

This is called reduction formula for gamma function.

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= n(n-1)(n-2)(n-3)\dots \Gamma(1)$$

$$= n! \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= n! \int_0^{\infty} e^{-x} dx$$

$$= n! \left[\frac{e^{-x}}{-1} \right]_0^{\infty}$$

$$= n! \left\{ - (e^{-\infty} - e^{-0}) \right\}$$

$$= n! \left\{ - (0 - 1) \right\}$$

$$= n!$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(n+1) = n!$$

Value of $\Gamma(n)$

$$\Gamma(1) = \Gamma(0+1) = 0! = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1! = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2! = 2 \times 1 = 2$$

$$\Gamma(4) = \Gamma(3+1) = 3! = 3 \times 2 \times 1 = 6$$

$$\Gamma(5) = \Gamma(4+1) = 4! = 4 \times 3 \times 2 \times 1 = 24$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

When n is not a +ve integer, the relation $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ is used to define $\Gamma(n)$.

$$\text{When } n=0, \Gamma(0) = \frac{\Gamma(0+1)}{0} = \infty$$

$$n=-1, \Gamma(-1) = \frac{\Gamma(-1+1)}{-1}$$

$$= \frac{\Gamma(0)}{-1} = \frac{\infty}{-1} = \infty$$

Thus when n is ~~zero~~ zero or a negative integer, $\Gamma(n)$ is not defined.

Q. Evaluate $\Gamma\left(\frac{3}{2}\right)$

Sol. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right)$
 $= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

Q. Evaluate $\Gamma\left(\frac{5}{2}\right)$

Sol. $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2} + 1\right)$
 $= \frac{3}{2} \Gamma\left(\frac{3}{2}\right)$
 $= \frac{3}{2} \Gamma\left(\frac{1}{2} + 1\right)$
 $= \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$
 $= \frac{3}{4} \sqrt{\pi}$

Q. $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2} + 1\right)$
 $= \frac{5}{2} \Gamma\left(\frac{5}{2}\right)$
 $= \frac{5}{2} \Gamma\left(\frac{3}{2} + 1\right)$
 $= \frac{5}{2} \cdot \frac{3}{2} \Gamma\left(\frac{3}{2}\right)$
 $= \frac{15}{4} \Gamma\left(\frac{1}{2} + 1\right)$
 $= \frac{15}{4} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$
 $= \frac{15}{8} \sqrt{\pi}$

Q. Evaluate $\Gamma\left(\frac{7}{2}\right), \Gamma\left(\frac{5}{2}\right), \Gamma\left(\frac{3}{2}\right), \Gamma\left(\frac{1}{2}\right)$

Sol.

~~$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}\right)$~~

$$\begin{aligned}\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2} + 1\right) \\ &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}}{2}\end{aligned}$$

$$\begin{aligned}\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) \\ &= \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}\end{aligned}$$

$$\begin{aligned}\Gamma\left(\frac{7}{2}\right) &= \Gamma\left(\frac{5}{2} + 1\right) = \frac{5}{2} \Gamma\left(\frac{5}{2}\right) \\ &= \frac{5}{2} \cdot \frac{3\sqrt{\pi}}{4} = \frac{15\sqrt{\pi}}{8}\end{aligned}$$

$$\Gamma\left(\frac{7}{2}\right), \Gamma\left(\frac{5}{2}\right), \Gamma\left(\frac{3}{2}\right), \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15\sqrt{\pi}}{8}, \frac{3\sqrt{\pi}}{4}, \frac{\sqrt{\pi}}{2}, \sqrt{\pi} = \frac{45\pi^2}{64}$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \quad (n > 0)$$

$$\Gamma(n+1) = n! \qquad \Gamma(0) = \infty$$

$$\Gamma(n+1) = n\Gamma(n) \qquad \Gamma(-1) = \infty$$

$$\cancel{\Gamma(n)} = \frac{\Gamma(n+1)}{n} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Q. Evaluate $\Gamma\left(-\frac{1}{2}\right)$

Sol.ⁿ

$$\begin{aligned} \Gamma\left(-\frac{1}{2}\right) &= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) \\ &= -2\sqrt{\pi} \end{aligned}$$

Q. Evaluate $\Gamma\left(-\frac{3}{2}\right)$

Sol.ⁿ

$$\begin{aligned} \Gamma\left(-\frac{3}{2}\right) &= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) \\ &= -\frac{2}{3} \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} \\ &= \frac{4}{3}\Gamma\left(\frac{1}{2}\right) = \frac{4}{3}\sqrt{\pi} \end{aligned}$$

Q. Evaluate $\Gamma\left(-\frac{5}{2}\right)$

Sol.ⁿ

$$\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right)$$

$$\begin{aligned}
&= -\frac{2}{5} \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{15} \Gamma\left(-\frac{1}{2}\right) \\
&= \frac{4}{15} \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{15} \Gamma\left(\frac{1}{2}\right) \\
&= -\frac{8}{15} \sqrt{\pi}
\end{aligned}$$

Q. Evaluate $\Gamma\left(-\frac{5}{2}\right)$, $\Gamma\left(-\frac{3}{2}\right)$, $\Gamma\left(-\frac{1}{2}\right)$

Sol.ⁿ

$$\begin{aligned}
\Gamma\left(-\frac{1}{2}\right) &= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2 \Gamma\left(\frac{1}{2}\right) \\
&= -2 \sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}
\Gamma\left(-\frac{3}{2}\right) &= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3} \Gamma\left(-\frac{1}{2}\right) \\
&= -\frac{2}{3} (-2 \sqrt{\pi}) = \frac{4}{3} \sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}
\Gamma\left(-\frac{5}{2}\right) &= \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5} \Gamma\left(-\frac{3}{2}\right) \\
&= -\frac{2}{5} \cdot \frac{4}{3} \sqrt{\pi} = -\frac{8}{15} \sqrt{\pi}
\end{aligned}$$

$$\therefore \Gamma\left(-\frac{5}{2}\right) \cdot \Gamma\left(-\frac{3}{2}\right) \cdot \Gamma\left(-\frac{1}{2}\right)$$

$$= -\frac{8}{15} \sqrt{\pi} \cdot \frac{4}{3} \sqrt{\pi} \cdot (-2 \sqrt{\pi})$$

$$= \frac{64}{45} \pi \sqrt{\pi}$$

Laplace Transforms

Let $f(t)$ be a function of t defined for all positive values of t . Then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ is defined

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

provided that the integral exists. s is a parameter which may be a real or complex number.

Clearly, $L\{f(t)\}$ being a function of s , is briefly written as $\bar{f}(s)$

$$\begin{aligned} \text{ie, } L\{f(t)\} &= \bar{f}(s) \\ \Rightarrow f(t) &= L^{-1}\{\bar{f}(s)\} \end{aligned}$$

Then $f(t)$ is called the inverse Laplace transform of $\bar{f}(s)$. The symbol L , which transforms $f(t)$ into $\bar{f}(s)$ is called Laplace transformation operator,

Formulae

$$(1) \quad L(1) = \frac{1}{s}$$

$$(2) \quad L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

$$(3) \quad L(e^{at}) = \frac{1}{s-a}$$

$$(4) \quad L(\sin at) = \frac{a}{s^2+a^2}$$

$$(5) \quad L(\cos at) = \frac{s}{s^2+a^2}$$

$$e^{-\infty} = 0$$

$$(6) \quad L(\sinh at) = \frac{a}{s^2-a^2}$$

$$x^0 = 1$$

$$(7) \quad L(\cosh at) = \frac{s}{s^2-a^2}$$

$$(1) \quad \text{Proof} \quad \text{L.H.S.} = L(1)$$

$$1) \int e^{ax} \sin bx \, dx = \int_0^{\infty} e^{-st} \cdot 1 \cdot dt$$
$$= \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) = \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$2) \int e^{ax} \cos bx \, dx = -\frac{1}{s} (e^{-\infty} - e^0)$$
$$= \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) = -\frac{1}{s} (0 - 1)$$
$$= \frac{1}{s} = \text{R.H.S.}$$

② Proof L.H.S. = $L(t^n)$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-st} t^n dt \quad \text{let } st = u \\
 & \qquad \qquad \qquad \frac{d(st)}{dt} = \frac{du}{dt} \\
 &= \int_0^{\infty} e^{-u} \cdot \left(\frac{u}{s}\right)^n \cdot \frac{du}{s} \Rightarrow s = \frac{du}{dt} \\
 & \qquad \qquad \qquad \Rightarrow dt = \frac{du}{s} \\
 &= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-u} \cdot u^n du \quad \begin{array}{l} t \rightarrow \infty, u \rightarrow \infty \\ t \rightarrow 0, u \rightarrow 0 \end{array} \\
 &= \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} = \text{R.H.S.}
 \end{aligned}$$

③ Proof L.H.S. = $L(e^{at})$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-st} \cdot e^{at} dt \\
 &= \int_0^{\infty} e^{-st+at} dt \\
 &= \int_0^{\infty} e^{-(s-a)t} dt \\
 &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \\
 &= -\frac{1}{s-a} [e^{-\infty} - e^0] \\
 &= -\frac{1}{s-a} (0 - 1) \\
 &= \frac{1}{s-a} = \text{R.H.S.}
 \end{aligned}$$

(4) Proof R.H.S. = $L(\sin at)$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-st} \sin at \, dt \\
 &= \left[\frac{e^{-st}}{(-s)^2 + a^2} (-s \sin at - a \cos at) \right]_0^{\infty} \\
 &= \left[-\frac{e^{-st}}{s^2 + a^2} (s \sin at + a \cos at) \right]_0^{\infty} \\
 &= -\frac{1}{s^2 + a^2} \left[e^{-\infty} (s \sin \infty + a \cos \infty) \right. \\
 &\quad \left. - e^0 (s \sin 0 + a \cos 0) \right] \\
 &= -\frac{1}{s^2 + a^2} [0 - 1 \cdot (0 + a)] \\
 &= \frac{a}{s^2 + a^2} = \text{R.H.S.} \quad \left[\begin{array}{l} \sinh ax = \frac{e^{ax} - e^{-ax}}{2} \\ \cosh ax = \frac{e^{ax} + e^{-ax}}{2} \end{array} \right]
 \end{aligned}$$

(6) Proof R.H.S. = $L(\sinh at)$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-st} \sinh at \, dt \\
 &= \int_0^{\infty} e^{-st} \cdot \frac{e^{at} - e^{-at}}{2} \, dt \\
 &= \frac{1}{2} \int_0^{\infty} (e^{-st+at} - e^{-st-at}) \, dt \\
 &= \frac{1}{2} \int_0^{\infty} \{ e^{-(s-a)t} - e^{-(s+a)t} \} \, dt
 \end{aligned}$$

$$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= \frac{1}{2} \left[\frac{e^{-\infty} - e^0}{-(s-a)} - \frac{e^{-\infty} - e^0}{-(s+a)} \right]$$

$$= \frac{1}{2} \left[\frac{0 - 1}{-(s-a)} - \frac{0 - 1}{-(s+a)} \right]$$

$$= \frac{1}{2} \left[\frac{-1}{-(s-a)} - \frac{-1}{-(s+a)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \cdot \frac{s+a - s+a}{s^2 - a^2}$$

$$= \frac{1}{2} \cdot \frac{2a}{s^2 - a^2}$$

$$= \frac{a}{s^2 - a^2} = \text{R.H.S.}$$

Properties of Laplace Transforms

① Linearity Property

Let a, b, c be any constants and f, g, h any functions of t , then

$$L\{a f(t) + b g(t) - c h(t)\} \\ = a L\{f(t)\} + b L\{g(t)\} - c L\{h(t)\}$$

Proof

$$\begin{aligned} \text{L.H.S.} &= L\{a f(t) + b g(t) - c h(t)\} \\ &= \int_0^{\infty} e^{-st} \{a f(t) + b g(t) - c h(t)\} dt \\ &= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt \\ &\quad - c \int_0^{\infty} e^{-st} h(t) dt \\ &= a L\{f(t)\} + b L\{g(t)\} - c L\{h(t)\} \\ &= \text{R.H.S.} \end{aligned}$$

Proved

Q. Find Laplace transform of
 $1 + 2t^3 - 4e^{3t} + 5e^{-t}$

Sol.ⁿ

$$\begin{aligned} & L(1 + 2t^3 - 4e^{3t} + 5e^{-t}) \\ &= L(1) + 2L(t^3) - 4L(e^{3t}) + 5L(e^{-t}) \\ &= \frac{1}{s} + 2 \cdot \frac{3!}{s^{3+1}} - 4 \cdot \frac{1}{s-3} + 5 \cdot \frac{1}{s-(-1)} \\ &= \frac{1}{s} + \frac{12}{s^4} - \frac{4}{s-3} + \frac{5}{s+1} \end{aligned}$$

Q. Find $L(3 \cosh 4t + 4 \sin 3t)$

Sol.ⁿ

$$\begin{aligned} & L(3 \cosh 4t + 4 \sin 3t) \\ &= 3L(\cosh 4t) + 4L(\sin 3t) \\ &= 3 \cdot \frac{s}{s^2 - 4^2} + 4 \cdot \frac{3}{s^2 + 3^2} \\ &= \frac{3s}{s^2 - 16} + \frac{12}{s^2 + 9} \end{aligned}$$

Q. Find $L(\sinh 5t - 5 \cos 4t)$

Sol.ⁿ

$$\begin{aligned} & L(\sinh 5t - 5 \cos 4t) \\ &= L(\sinh 5t) - 5L(\cos 4t) \\ &= \frac{5}{s^2 - 5^2} - 5 \cdot \frac{s}{s^2 + 4^2} \\ &= \frac{5}{s^2 - 25} - \frac{5s}{s^2 + 16} \end{aligned}$$

Q. find $L(\sqrt{t})$

Sol. $L(\sqrt{t}) = L(t^{1/2})$

$$= \frac{\Gamma(\frac{1}{2} + 1)}{s^{\frac{1}{2} + 1}}$$
$$= \frac{\frac{1}{2} \Gamma(\frac{1}{2})}{s^{3/2}} = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

Q. find $L\left(\frac{1}{\sqrt{t}}\right)$ ~~$L(t^{-1/2})$~~

Sol. $L\left(\frac{1}{\sqrt{t}}\right) = L(t^{-1/2})$

$$= \frac{\Gamma(-\frac{1}{2} + 1)}{s^{-\frac{1}{2} + 1}}$$
$$= \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

Q. find $L(t^{-3/2})$

Sol. $L(t^{-3/2}) = \frac{\Gamma(-\frac{3}{2} + 1)}{s^{-\frac{3}{2} + 1}} = \frac{\Gamma(-\frac{1}{2})}{s^{-1/2}}$

$$= s^{1/2} \cdot \frac{\Gamma(-\frac{1}{2} + 1)}{-\frac{1}{2} + 1} = \sqrt{s} \cdot \frac{\Gamma(\frac{1}{2})}{\frac{1}{2}}$$
$$= 2\sqrt{s} \sqrt{\pi} = 2\sqrt{\pi s}$$

Q. Find $L\{(s \sin t - \cos t)^2\}$

Sol.ⁿ $(s \sin t - \cos t)^2 = s^2 \sin^2 t + \cos^2 t - 2 s \sin t \cdot \cos t$
 $= 1 - s^2 \sin 2t$

$$\begin{aligned}\therefore L\{(s \sin t - \cos t)^2\} &= L(1 - s^2 \sin 2t) \\ &= L(1) - L(s^2 \sin 2t) \\ &= \frac{1}{s} - \frac{s^2}{s^2 + 2^2} \\ &= \frac{1}{s} - \frac{s^2}{s^2 + 4} \\ &= \frac{s^2 + 4 - s^2}{s(s^2 + 4)}\end{aligned}$$

Q. Find $L\{\cos(at+b)\}$

Sol.ⁿ $L\{\cos(at+b)\} = L(\cos at \cdot \cos b - \sin at \cdot \sin b)$
 $= \cos b \cdot L(\cos at) - \sin b \cdot L(\sin at)$
 $= \cos b \cdot \frac{s}{s^2 + a^2} - \sin b \cdot \frac{a}{s^2 + a^2}$
 $= \frac{s \cos b - a \sin b}{s^2 + a^2}$

~~Q. Find~~ $\cos 2A = 2 \cos^2 A - 1$

$$\Rightarrow 2 \cos^2 A = 1 + \cos 2A$$

$$\Rightarrow \cos^2 A = \frac{1 + \cos 2A}{2}$$

Q. Find $L(\cos at \cdot \cos bt)$

Sol.ⁿ $\cos at \cdot \cos bt = \frac{1}{2} \cdot 2 \cos at \cdot \cos bt$
 $= \frac{1}{2} \{ \cos(at+bt) + \cos(at-bt) \}$
 $= \frac{1}{2} \{ \cos(a+b)t + \cos(a-b)t \}$

$$L(\cos at \cdot \cos bt) = L \left[\frac{1}{2} \{ \cos(a+b)t + \cos(a-b)t \} \right]$$
$$= \frac{1}{2} \left[L \{ \cos(a+b)t \} + L \{ \cos(a-b)t \} \right]$$
$$= \frac{1}{2} \left\{ \frac{s}{s^2 + (a+b)^2} + \frac{s}{s^2 + (a-b)^2} \right\}$$
$$= \frac{s}{2} \cdot \frac{s^2 + (a-b)^2 + s^2 + (a+b)^2}{\{s^2 + (a+b)^2\} \{s^2 + (a-b)^2\}}$$
$$= \frac{s}{2} \cdot \frac{2s^2 + 2a^2 + 2b^2}{\{s^2 + (a+b)^2\} \{s^2 + (a-b)^2\}}$$
$$= \frac{s(s^2 + a^2 + b^2)}{\{s^2 + (a+b)^2\} \{s^2 + (a-b)^2\}}$$

Q. Find $L(\cos^2 2t)$

Sol.ⁿ $\cos^2 2t = \frac{1}{2} \cdot 2 \cos^2 2t$
 $= \frac{1 + \cos 4t}{2}$

$$L(\cos^2 2t) = L \left(\frac{1 + \cos 4t}{2} \right) = \frac{1}{2} \{ L(1) + L(\cos 4t) \}$$
$$= \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 + 4^2} \right) = \frac{1}{2} \cdot \frac{s^2 + 16 + s^2}{s(s^2 + 16)}$$

$$= \frac{s^2 + 8}{s(s^2 + 16)}$$

Q. Find $L(\sin^3 2t)$

Sol. $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\Rightarrow 4 \sin^3 A = 3 \sin A - \sin 3A$$

$$\Rightarrow \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A$$

$$\begin{aligned} L(\sin^3 2t) &= L\left(\frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t\right) \\ &= \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t) \\ &= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{6}{s^2 + 6^2} \\ &= \frac{3}{2} \left(\frac{1}{s^2 + 4} - \frac{1}{s^2 + 36} \right) \\ &= \frac{3}{2} \cdot \frac{s^2 + 36 - s^2 - 4}{(s^2 + 4)(s^2 + 36)} \\ &= \frac{3}{2} \cdot \frac{32}{(s^2 + 4)(s^2 + 36)} \\ &= \frac{48}{(s^2 + 4)(s^2 + 36)} \end{aligned}$$

Q. Find $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\}$

Q. Find $L(\sin^2 3t)$

Q. Find $L(\cos^3 4t)$

$$f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

Sol.ⁿ $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$

$$= \int_0^1 e^{-st} \cdot e^t dt + \int_1^{\infty} e^{-st} \cdot 0 dt$$

$$= \int_0^1 e^{-st+t} dt$$

$$= \int_0^1 e^{(1-s)t} dt$$

$$= \left[\frac{e^{(1-s)t}}{1-s} \right]_0^1$$

$$= \frac{e^{1-s} - e^0}{1-s} = \frac{e^{1-s} - 1}{1-s}$$

② First Shifting Property

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at} f(t)\} = \bar{f}(s-a)$$

Proof L.H.S. = $L\{e^{at} f(t)\}$

$$= \int_0^{\infty} e^{-st} \cdot e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-st+at} f(t) dt$$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-(s-a)t} f(t) dt \\
 & \qquad \qquad \qquad \text{Put } s-a=r \\
 &= \int_0^{\infty} e^{-rt} f(t) dt \\
 &= \bar{f}(r) = \bar{f}(s-a) = \text{R.H.S.}
 \end{aligned}$$

formulae

- ① $\therefore L(1) = \frac{1}{s} \qquad \therefore L(e^{at}) = \frac{1}{s-a}$
- ② $\therefore L(t^n) = \frac{n!}{s^{n+1}} \qquad \therefore L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$
- ③ $\therefore L(\sin bt) = \frac{b}{s^2+b^2} \qquad \therefore L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$
- ④ $\therefore L(\cos bt) = \frac{s}{s^2+b^2} \qquad \therefore L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$
- ⑤ $\therefore L(\sinh bt) = \frac{a}{s^2-b^2} \qquad \therefore L(e^{at}\sinh bt) = \frac{a}{(s-a)^2-b^2}$
- ⑥ $\therefore L(\cosh bt) = \frac{s}{s^2-b^2} \qquad \therefore L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2-b^2}$

Q. find $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

Sol. $\therefore L(2\cos 5t - 3\sin 5t)$

$$\begin{aligned}
 &= 2L(\cos 5t) - 3L(\sin 5t) \\
 &= 2 \cdot \frac{s}{s^2+5^2} - 3 \cdot \frac{5}{s^2+5^2} \\
 &= \frac{2s - 15}{s^2 + 25}
 \end{aligned}$$

$$\begin{aligned}
\therefore L \{ e^{-3t} (2 \cos 5t - 3 \sin 5t) \} \\
&= \frac{2(s - (-3)) - 15}{(s - (-3))^2 + 25} \\
&= \frac{2(s + 3) - 15}{(s + 3)^2 + 25} \\
&= \frac{2s + 6 - 15}{s^2 + 6s + 9 + 25} \\
&= \frac{2s - 9}{s^2 + 6s + 34}
\end{aligned}$$

$$\begin{aligned}
\cos 2A \\
&= 2 \cos^2 A - 1 \\
2 \cos^2 A &= 1 + \cos 2A \\
\cos^2 A &= \frac{1 + \cos 2A}{2}
\end{aligned}$$

Q. Find $L(e^{2t} \cos^2 t)$

Sol. $\therefore L(\cos^2 t) = L\left(\frac{1 + \cos 2t}{2}\right)$

$$\begin{aligned}
&= \frac{1}{2} \{ L(1) + L(\cos 2t) \} \\
&= \frac{1}{2} \left\{ \frac{1}{s} + \frac{s}{s^2 + 2^2} \right\} \\
&= \frac{1}{2} \cdot \frac{s^2 + 4 + s^2}{s(s^2 + 4)} \\
&= \frac{s^2 + 2}{s(s^2 + 4)}
\end{aligned}$$

$$\begin{aligned}
\therefore L(e^{2t} \cos^2 t) &= \frac{(s - 2)^2 + 2}{(s - 2) \{ (s - 2)^2 + 4 \}} \\
&= \frac{s^2 - 4s + 4 + 2}{(s - 2) \{ s^2 - 4s + 4 + 4 \}} \\
&= \frac{s^2 - 4s + 6}{(s - 2)(s^2 - 4s + 8)}
\end{aligned}$$

Transforms of Derivatives

Let $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$

Then $L\{f'(t)\} = s\bar{f}(s) - f(0)$,

Proof

$$\text{L.H.S.} = L\{f'(t)\}$$

$$= \int_0^{\infty} e^{-st} f'(t) dt$$

$$= \left[e^{-st} \int f'(t) dt - \int (e^{-st})' \left\{ \int f'(t) dt \right\} dt \right]_0^{\infty}$$

$$= \left[e^{-st} f(t) - \int e^{-st} \cdot (-s) \cdot f(t) dt \right]_0^{\infty}$$

$$= e^{-\infty} f(\infty) - e^0 f(0) + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= 0 - f(0) + s L\{f(t)\}$$

$$= s\bar{f}(s) - f(0)$$

$$= \text{R.H.S.}$$

Proved

Transforms of Integrals

Let $L\{f(t)\} = \bar{f}(s)$, then $L\left(\int_0^t f(u) du\right) = \frac{1}{s} \bar{f}(s)$

Q. Find $L\left(\int_0^t e^{-t} \cos t dt\right)$

Sol. $\because L(\cos t) = \frac{s}{s^2 + 1}$

$$\therefore L(e^{-t} \cos t) = \frac{s - (-1)}{(s - (-1))^2 + 1}$$

$$= \frac{s + 1}{(s + 1)^2 + 1}$$

$$\therefore L\left(\int_0^t e^{-t} \cos t \, dt\right) = \frac{1}{s} \cdot \frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s(s^2+2s+2)}$$

Multiplication by t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \{\bar{f}(s)\}$$

where $n = 1, 2, 3, \dots$

Q. Find $L(t \cos at)$

Solⁿ $\therefore L(\cos at) = \frac{s}{s^2+a^2}$

$$L(t \cos at) = (-1)^1 \frac{d^1}{ds^1} \left(\frac{s}{s^2+a^2} \right)$$

$$= - \frac{1 \cdot (s^2+a^2) - 2s \cdot s}{(s^2+a^2)^2}$$

$$= - \frac{s^2+a^2 - 2s^2}{(s^2+a^2)^2}$$

$$= - \frac{-s^2+a^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

Q. Find $L(t^2 \sin at)$

Solⁿ $\therefore L(\sin at) = \frac{a}{s^2+a^2} = a (s^2+a^2)^{-1}$

$$L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2} \left\{ a (s^2+a^2)^{-1} \right\}$$

$$= \frac{d}{dx} [p(x) \cos(x^2 + 2x)]$$

$$= -2x \frac{d}{dx} [p(x) \cos(x^2 + 2x)]$$

$$= -2x \left[\frac{p'(x) \cos(x^2 + 2x) - p(x) \sin(x^2 + 2x) \cdot (2x + 2)}{\cos^2(x^2 + 2x)} \right]$$

$$= -2x \frac{p'(x) \cos(x^2 + 2x) - 2x p(x) \sin(x^2 + 2x) - 2p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$= -2x \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$= -2x \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$= -2x \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$\frac{d}{dx} [p(x) \cos(x^2 + 2x)] = \frac{d}{dx} p(x) \cos(x^2 + 2x) - p(x) \frac{d}{dx} \cos(x^2 + 2x)$$

$$= p'(x) \cos(x^2 + 2x) - p(x) \frac{d}{dx} \cos(x^2 + 2x)$$

$$= p'(x) \cos(x^2 + 2x) - p(x) \sin(x^2 + 2x) \cdot (2x + 2)$$

$$= \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$\frac{d}{dx} [p(x) \cos(x^2 + 2x)] = \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$= \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

$$= \frac{p'(x) \cos(x^2 + 2x) - 2(x+1)p(x) \sin(x^2 + 2x)}{\cos^2(x^2 + 2x)}$$

Division by t

∴ $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s) ds$
provided the integral exists.

Q. Find $L\left(\frac{1-e^t}{t}\right)$

Sol. ∴ $L(1-e^t) = L(1) - L(e^t)$
 $= \frac{1}{s} - \frac{1}{s-1}$

$$\therefore L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \left[\ln s - \ln(s-1) \right]_s^\infty$$

$$= \left[\ln \frac{s}{s-1} \right]_s^\infty$$

$$= \left[\ln \frac{\frac{s}{s} - \frac{1}{s}}{\frac{s}{s} - \frac{1}{s}} \right]_s^\infty$$

$$= \left[\ln \frac{1}{1 - \frac{1}{s}} \right]_s^\infty$$

$$= \ln \frac{1}{1-0} - \ln \frac{1}{1 - \frac{1}{s}}$$

$$= \ln 1 + \ln \left(\frac{s}{s-1}\right)^{-1}$$

$$= 0 + \ln \frac{s-1}{s}$$

$$= \ln \frac{s-1}{s}$$

$$Q. \text{ Find } L \left(\frac{\cos at - \cos bt}{t} \right)$$

$$\underline{\text{Sol.}^n} \quad \therefore L(\cos at - \cos bt) = L(\cos at) - L(\cos bt)$$

$$= \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2}$$

$$\therefore L \left(\frac{\cos at - \cos bt}{t} \right) = \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds$$

$$= \frac{1}{2} \int_s^\infty \left(\frac{2s}{s^2 + a^2} - \frac{2s}{s^2 + b^2} \right) ds$$

$$= \frac{1}{2} \left[\ln(s^2 + a^2) - \ln(s^2 + b^2) \right]_s^\infty$$

$$= \frac{1}{2} \left[\ln \frac{s^2 + a^2}{s^2 + b^2} \right]_s^\infty$$

$$= \frac{1}{2} \left[\ln \frac{1 + a^2/s^2}{1 + b^2/s^2} \right]_s^\infty$$

$$= \frac{1}{2} \left[\ln \frac{1 + \frac{a^2}{\infty^2}}{1 + \frac{b^2}{\infty^2}} - \ln \frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}} \right]$$

$$= \frac{1}{2} \left[\ln \frac{1 + 0}{1 + 0} - \ln \frac{s^2 + a^2}{s^2 + b^2} \right]$$

$$= \frac{1}{2} \left[\ln 1 + \ln \left(\frac{s^2 + a^2}{s^2 + b^2} \right)^{-1} \right]$$

$$= \frac{1}{2} \left[0 + \ln \frac{s^2 + b^2}{s^2 + a^2} \right]$$

$$= \frac{1}{2} \ln \frac{s^2 + b^2}{s^2 + a^2}$$

Inverse Laplace Transforms

Formulae

$$1) L^{-1}\left(\frac{1}{s}\right) = 1$$

$$13) L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin at$$

$$2) L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$14) L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$3) L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$$

$$= \frac{1}{2a^3} (\sin at - at \cos at)$$

Where $n = 1, 2, 3, \dots$

$$4) L^{-1}\frac{1}{(s-a)^n} = \frac{e^{at} t^{n-1}}{(n-1)!}$$

$$5) L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$$

$$6) L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$7) L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a} \sinh at$$

$$8) L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$9) L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at} \sin bt$$

$$10) L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at} \cos bt$$

$$11) L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b} e^{at} \sinh bt$$

$$12) L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at} \cosh bt$$

Q. Find the inverse transforms of

$$\frac{s^2 - 3s + 4}{s^3}$$

Sol. $L^{-1} \left(\frac{s^2 - 3s + 4}{s^3} \right)$

$$= L^{-1} \left(\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3} \right)$$

$$= L^{-1} \left(\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3} \right)$$

$$= L^{-1} \left(\frac{1}{s} \right) - 3 L^{-1} \left(\frac{1}{s^2} \right) + 4 L^{-1} \left(\frac{1}{s^3} \right)$$

$$= 1 - 3 \frac{t^{2-1}}{(2-1)!} + 4 \frac{t^{3-1}}{(3-1)!}$$

$$= 1 - \frac{3t}{1} + 4 \cdot \frac{t^2}{2}$$

$$= 1 - 3t + 2t^2$$

Q. Find $L^{-1} \left(\frac{s+2}{s^2 - 4s + 13} \right)$

Sol. $L^{-1} \left(\frac{s+2}{s^2 - 4s + 13} \right)$

$$= L^{-1} \left\{ \frac{(s-2)+4}{(s-2)^2 + 9} \right\}$$

$$\begin{aligned} s^2 - 4s + 13 &= s^2 - 2 \cdot s \cdot 2 + 2^2 \\ &\quad - 2^2 + 13 \\ &= (s-2)^2 + 9 \end{aligned}$$

$$= L^{-1} \left\{ \frac{s-2}{(s-2)^2 + 9} \right\} + 4 L^{-1} \left\{ \frac{1}{(s-2)^2 + 9} \right\}$$

$$= e^{2t} \cos 3t + 4 \cdot L^{-1} \left\{ \frac{1}{(s-2)^2 + 9} \right\}$$

$$= e^{2t} \left(\cos 3t + \frac{4}{3} \sin 3t \right)$$

Q. find $L^{-1} \left\{ \frac{s^2 + s - 2}{s(s+3)(s-2)} \right\}$

Sol.ⁿ Let $\frac{s^2 + s - 2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$ — (1)

$$\Rightarrow s^2 + s - 2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3)$$

Put $s=2$, $2^2 + 2 - 2 = C \cdot 2(2+3)$

$$\Rightarrow 4 = 10C \Rightarrow \boxed{C = \frac{2}{5}}$$

Put $s=-3$, $(-3)^2 + (-3) - 2 = B(-3)(-3-2)$

$$\Rightarrow 9 - 3 - 2 = 15B$$

$$\Rightarrow 4 = 15B \Rightarrow \boxed{B = \frac{4}{15}}$$

Put $s=0$, $0 + 0 - 2 = A(0+3)(0-2)$

$$\Rightarrow -2 = -6A \Rightarrow \boxed{A = \frac{1}{3}}$$

Putting the values of A, B and C in (1),

$$\frac{s^2 + s - 2}{s(s+3)(s-2)} = \frac{1/3}{s} + \frac{4/15}{s+3} + \frac{2/5}{s-2}$$

On inversion,

$$\begin{aligned} L^{-1} \left\{ \frac{s^2 + s - 2}{s(s+3)(s-2)} \right\} &= \frac{1}{3} L^{-1} \left(\frac{1}{s} \right) + \frac{4}{15} L^{-1} \left(\frac{1}{s+3} \right) \\ &\quad + \frac{2}{5} L^{-1} \left(\frac{1}{s-2} \right) \\ &= \frac{1}{3} \cdot 1 + \frac{4}{15} e^{-3t} + \frac{2}{5} e^{2t} \\ &= \frac{1}{3} + \frac{4e^{-3t}}{15} + \frac{2e^{2t}}{5} \end{aligned}$$

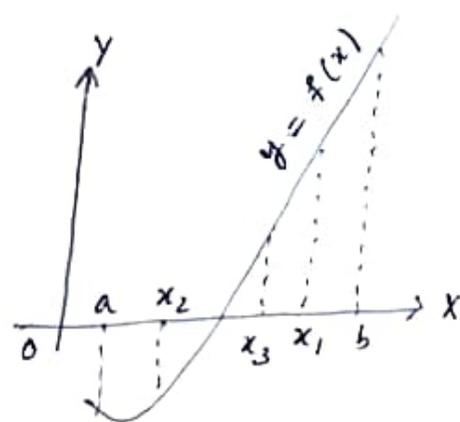
Q. find $L^{-1} \left\{ \frac{4s+5}{(s-1)^2(s+2)} \right\}$

Sol.ⁿ Let $\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$

Numerical Methods

① Bisection Method

This method consists in locating the root of the equation $f(x) = 0$ between a and b . If $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b .



Then the first approximation to the root is $x_1 = \frac{a+b}{2}$.

Now, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative.

Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

Q. Find a root of the equation $x^3 - 4x - 9 = 0$ using the bisection method in four stages.

Solⁿ. Let $f(x) = x^3 - 4x - 9$

$$\therefore f(2) = 8 - 8 - 9 = -9 \text{ is } -ve$$

$$\text{and } f(3) = 27 - 12 - 9 = 6 \text{ is } +ve$$

\therefore The root lies between 2 and 3.

Hence the first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

$$\text{Now, } f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375 \text{ is } -ve.$$

\therefore The root lies between $x_1 = 2.5$ and 3.

Hence the second approximation to the root is

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$\text{Now, } f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \text{ is } +ve$$

∴ The root lies between $x_1 = 2.5$ and $x_2 = 2.75$

Hence the 3rd approximation to the root is

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

Now, $f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ is -ve.

∴ The root lies between $x_3 = 2.625$ and $x_2 = 2.75$

Hence the 4th approximation to the root is

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

(2) Method of false position (OR Regula-falsi method)

Let the real root of an equation $y = f(x) = 0$.

Choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs.

Equation of the chord joining the points $A[x_0, f(x_0)]$ and

$B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \quad \text{--- (1)}$$

The method consists in replacing the curve AB by the chord AB and taking the point of intersection of the chord with the x -axis as an approximation to the root. $(x_2, 0)$ is the intersection point

$$\text{then } x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{--- (2)}$$

which is an approximation to the root.

Then next check $f(x_0)$ and $f(x_2)$ or $f(x_1)$ and $f(x_2)$ are opposite signs. This procedure is repeated till the root is bound to desired accuracy.

Q. Find the root of the equation $x^3 + x - 1 = 0$ near $x=1$ by the method of false position.

Sol. Let $f(x) = x^3 + x - 1$
∴ $f(0) = -1$ and $f(1) = 1$

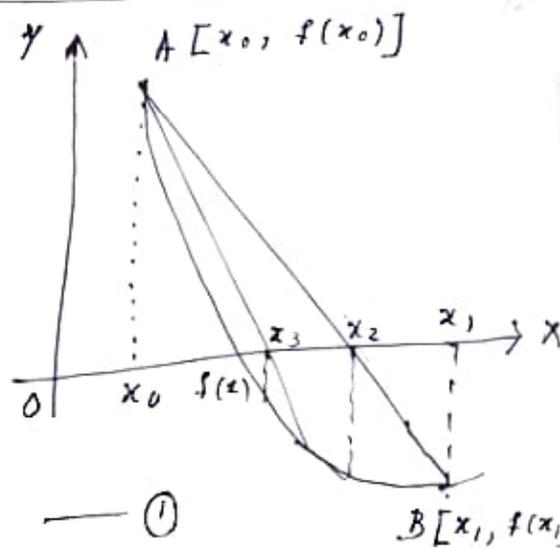
∵ $f(0)$ and $f(1)$ have opposite sign, the root lies between $x_0 = 0$ and $x_1 = 1$.

∴ The 1st approximation of the root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1 - 0}{1 - (-1)} (-1) = 0.5$$

$$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375$$



∴ $f(0.5)$ and $f(1)$ have opposite sign, the root lies between $x_0 = 0.5$ and $x_1 = 1$

∴ The 2nd approximation of the root is

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 0.5 - \frac{1 - 0.5}{1 - (-0.375)} (-0.375) = 0.64$$

$$f(0.64) = (0.64)^3 + 0.64 - 1 = -0.0979$$

∴ $f(0.64)$ and $f(1)$ have opposite sign, the root lies between $x_0 = 0.64$ and $x_1 = 1$

∴ The 3rd approximation of the root is

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 0.64 - \frac{1 - 0.64}{1 - (-0.0979)} (-0.0979) = 0.672$$

Hence the root of $f(x) = 0$ near 1 is 0.672.

③ Newton-Raphson Method (OR Newton's iteration formula)

Let x_0 be an approximate root of the equation $f(x) = 0$, let $x_1 = x_0 + h$ be the exact root, then

$$f(x_1) = 0$$

$$\Rightarrow f(x_0 + h) = 0$$

$$\Rightarrow f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0 \quad \left[\text{by Taylor's series} \right]$$

∵ h is small, neglecting h^2 and higher powers of h

$$\text{We get, } f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = - \frac{f(x_0)}{f'(x_0)} \quad \text{--- (1)}$$

∴ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

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Similarly, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Q. Find the real root of the equation $x^4 - x - 9 = 0$ by Newton-Raphson method, correct to three places of decimal.

Solⁿ Let $f(x) = x^4 - x - 9$ and $f'(x) = 4x^3 - 1$
 $f(1) = -9$ (-ve) and $f(2) = 5$ (+ve)

∵ $f(1)$ and $f(2)$ are of opposite signs, the real root of $f(x) = 0$ lies between 1 and 2

Also $f(2) < f(1)$ numerically, we take the 1st approximation $x_1 = 2$

The 2nd approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^4 - x_1 - 9}{4x_1^3 - 1} = \frac{3x_1^4 + 9}{4x_1^3 - 1}$$

$$= \frac{3(2^4) + 9}{4(2^3) - 1} = \frac{57}{31} = 1.8$$

The 3rd approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3x_2^4 + 9}{4x_2^3 - 1} = \frac{3(1.8)^4 + 9}{4(1.8)^3 - 1} = 1.81$$

The 4th approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{3x_3^4 + 9}{4x_3^3 - 1} = \frac{3(1.81)^4 + 9}{4(1.81)^3 - 1}$$

$$= 1.813$$

Hence the real root of $f(x) = 0$, correct to three decimal places is 1.813.

Finite Differences and Interpolation

Finite differences

Let $y = f(x)$ be a discrete function of the independent variable x . When x assumes the values $x_0, x_1, x_2, \dots, x_n$, let the corresponding values of y be $y_0, y_1, y_2, \dots, y_n$ where n is a positive integer. The values of x are called the arguments and those of y are known as entries. The equal interval of the arguments, namely h is called the interval of differencing. The process of obtaining the value of y for any ~~in~~ intermediate value of x is termed as interpolation. If however we need a value of y for some value of x outside the given range $(a, a+nh)$, then the method of obtaining y is called extrapolation.

Types of differences

1. Forward difference, Δ
2. Backward difference, ∇
3. Central difference, δ

$$\Delta \{f(x)\} = f(x+h) - f(x)$$

Q. Evaluate $\Delta (\tan^{-1} x)$

Sol. $\Delta (\tan^{-1} x) = \tan^{-1}(x+h) - \tan^{-1} x$

$$\begin{cases} f(x) = \tan^{-1} x \\ f(x+h) = \tan^{-1}(x+h) \end{cases}$$

$$= \tan^{-1} \frac{x+h-x}{1+(x+h)x}$$

$$= \tan^{-1} \frac{h}{1+x^2+hx}$$

Q. Evaluate $\Delta (\sin x)$

Sol. $\Delta (\sin x) = \sin(x+h) - \sin x$

$$\begin{cases} f(x) = \sin x \\ f(x+h) = \sin(x+h) \end{cases}$$

$$= 2 \cos \frac{x+h+x}{2} \sin \frac{x+h-x}{2}$$

$$= 2 \cos \frac{2x+h}{2} \sin \frac{h}{2}$$

Q. Evaluate $\Delta (e^x)$

$$f(x) = e^x$$

Sol. $\Delta (e^x) = e^{x+h} - e^x$

$$f(x+h) = e^{x+h}$$

$$= e^x (e^h - 1)$$

Q. Evaluate $\Delta^n (e^x)$

interval of differencing being unity.

Sol.

$$\Delta (e^x) = e^{x+1} - e^x \quad f(x) = e^x$$
$$= e^x (e^1 - 1) = e^x (e-1) \quad f(x+h) = e^{x+1}$$

$$\Delta^2 (e^x) = \Delta \{ \Delta (e^x) \}$$
$$= \Delta \{ e^x (e-1) \}$$
$$= (e-1) \Delta (e^x)$$
$$= (e-1) (e^{x+1} - e^x)$$
$$= (e-1) \cdot e^x (e-1)$$
$$= (e-1)^2 e^x$$

$$\Delta^3 (e^x) = \Delta \{ \Delta^2 (e^x) \}$$
$$= \Delta \{ (e-1)^2 e^x \}$$
$$= (e-1)^2 \Delta (e^x)$$
$$= (e-1)^2 (e^{x+1} - e^x)$$
$$= (e-1)^2 \cdot e^x (e-1)$$
$$= (e-1)^3 \cdot e^x$$

$$\vdots$$
$$\Delta^n (e^x) = (e-1)^n e^x$$

Q. Form the forward difference table for the following data

x	0	1	2	3	4
y	8	11	9	15	6

Sol.ⁿ The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8	3	-5	13	-36
1	11	-2	8	-23	
2	9	6	-15		
3	15	-9			
4	6				

Q. Write the forward difference table, if

x	10	20	30	40
$f(x)$	1.1	2.0	4.5	7.9

Sol.ⁿ Evaluate $\Delta f(30)$, $\Delta^2 f(20)$, $\Delta^3 f(10)$
The forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
10	1.1	.9	1.6	
20	2.0	2.5	.9	-0.7
30	4.5	3.4		
40	7.9			

$$\Delta f(30) = 3.4$$

$$\Delta^2 f(20) = .9$$

$$\Delta^3 f(10) = -0.7$$

Q. Construct the forward difference table, if

x	0	1	2	3	4
$f(x)$	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta f(1)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$

Sol.ⁿ The forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1.0	.5	.2	0	.4
1	1.5	.7	.2	.4	
2	2.2	.9	.6		
3	3.1	1.5			
4	4.6				

$$\Delta f(1) = .7, \quad \Delta^2 f(0) = .2$$

$$\Delta^3 f(0) = 0$$

Q. Form the backward difference table of the function $f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$

and Evaluate $\nabla f(4)$, $\nabla^2 f(2)$, $\nabla^3 f(3)$, $\nabla^4 f(5)$

Sol.ⁿ

x	-1	0	1	2	3	4	5
$f(x)$	-6	-7	-14	-21	-22	-11	18

The backward difference table is

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
-1	-6	-1	-6		
0	-7	-7	0	6	0
1	-14	-7	6	6	0
2	-21	-1	6	6	0
3	-22	11	12	6	
4	-11	29	18		
5	18				

$$\nabla f(4) = 11 \quad \nabla^2 f(2) = 0$$

$$\nabla^3 f(3) = 6$$

$$\nabla^4 f(5) = 0$$

Q. Find the missing values of the following table:

x	0	1	2	3	4
y	1	3	9	-	81

Sol.ⁿ The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	2	4	$a - 19$	$124 - 4a$
1	3	6	$a - 15$	$105 - 3a$	
2	9	$a - 9$	$98 - 2a$		
3	a	$81 - a$			
4	81				

$$\therefore \Delta^4 y = 0$$

$$\Rightarrow 124 - 4a = 0$$

$$\Rightarrow 4a = 124$$

$$\Rightarrow \boxed{a = 31}$$

Hence the missing value is 31.

Operator

Shift operator E is the operation of increasing the argument x by h so that

$$E f(x) = f(x+h)$$

$$E^2 f(x) = f(x+2h)$$

$$E^3 f(x) = f(x+3h), \text{ etc.}$$

$$E^{-1} f(x) = f(x-h)$$

$$E^{-2} f(x) = f(x-2h), \text{ etc.}$$

$$E y_x = y_{x+h}$$

$$E^{-1} y_x = y_{x-h}$$

$$E \sin x = \sin(x+h)$$

$$E \log_e x = \log_e(x+h)$$

$$E e^x = e^{x+h}$$

Relation between the operators

$$\textcircled{1} \Delta = E - 1$$

Proof $\Delta f(x) = f(x+h) - f(x)$

$$= E f(x) - f(x)$$

$$= (E - 1) f(x)$$

$$\Rightarrow \Delta = E - 1$$

Proved

$$1 + \Delta = E$$

$$\textcircled{2} \nabla = 1 - E^{-1}$$

Proof $\nabla f(x) = f(x) - f(x-h)$

$$= f(x) - E^{-1} f(x)$$

$$= (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = 1 - E^{-1}$$

Proved

$$(3) \Delta = E \nabla = \nabla E$$

Proof $\nabla f(x) = f(x) - f(x-h)$

$$\begin{aligned} E \nabla f(x) &= E f(x) - E f(x-h) \\ &= f(x+h) - f(x+h-h) \\ &= f(x+h) - f(x) \\ &= \Delta f(x) \end{aligned}$$

$$\Rightarrow E \nabla = \Delta \quad \text{--- (i)}$$

$$E f(x) = f(x+h)$$

$$\begin{aligned} \nabla E f(x) &= \nabla f(x+h) \\ &= f(x+h) - f(x-h+h) \\ &= f(x+h) - f(x) \\ &= \Delta f(x) \end{aligned}$$

$$\Rightarrow \nabla E = \Delta \quad \text{--- (ii)}$$

From (i) and (ii),

$$\Delta = E \nabla = \nabla E \quad \text{--- Proved}$$

$$(4) (E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$$

Proof L.H.S. = $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2}$

$$= (E^{1/2} + E^{-1/2})E^{1/2}$$

$$= E + 1$$

$$= 1 + \Delta + 1$$

$$= 2 + \Delta = \text{R.H.S.} \quad \text{--- Proved}$$

Q. Evaluate $(2\Delta^2 + \Delta - 1)(x^2 + 2x + 1)$, taking $h=1$ as the interval of differencing.

Interpolation with unequal intervals

Lagrange's formula for unequal intervals →

If $y = f(x)$ takes the values y_0, y_1, \dots, y_n corresponding to $x = x_0, x_1, \dots, x_n$,

$$\begin{aligned} \text{then } f(x) &= \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \times y_0 \\ &+ \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} \times y_1 \\ &+ \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \times y_n \end{aligned}$$

Q. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

evaluate $f(9)$ by using Lagrange's formula

Sol.ⁿ

$$\begin{aligned} f(9) &= \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\ &+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\ &+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\ &+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\ &+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \end{aligned}$$

$$\begin{aligned}
&= \frac{2(-2)(-4)(-8) \times 150}{(-2)(-6)(-8)(-12)} + \frac{4(-2)(-4)(-8) \times 372}{2(-4)(-6)(-10)} \\
&+ \frac{4(2)(-4)(-8) \times 1452}{6(4)(-2)(-6)} + \frac{4(2)(-2)(-8) \times 2366}{8(6)(2)(-4)} \\
&+ \frac{4(2)(-2)(-4) \times 5202}{12(10)(6)(4)} \\
&= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} \\
&= 810
\end{aligned}$$

Q. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given

x	5	6	9	11
y	12	13	14	16

Solⁿ

$$\begin{aligned}
y(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 \\
&+ \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 \\
&+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 \\
&+ \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16 \\
&= \frac{4(1)(-1) \times 12}{(-1)(-4)(-6)} + \frac{5(1)(-1) \times 13}{1(-3)(-5)} \\
&+ \frac{5(4)(-1) \times 14}{4(3)(-2)} + \frac{5(4)(1) \times 16}{6(5)(2)}
\end{aligned}$$

$$= \frac{-48}{-24} + \frac{-65}{15} + \frac{-280}{-24} + \frac{320}{60}$$

$$= 2 - 4.33 + 11.66 + 5.33$$

$$= 14.66 \text{ (approx.)}$$

Inverse Interpolation

Q. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data:

x	5	6	9	11
$f(x)$	12	13	14	16

Sol.

$f(x)$	12	13	14	16
x	5	6	9	11

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5$$

$$+ \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} \times 6$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9$$

$$+ \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$= \frac{2(1)(-1) \times 5}{(-1)(-2)(-4)} + \frac{3(1)(-1) \times 6}{1(-1)(-3)} + \frac{3(2)(-1) \times 9}{2(1)(-2)}$$

$$+ \frac{3(2)(1) \times 11}{4(3)(2)} = \frac{5}{4} - 6 + \frac{27}{2} + \frac{11}{4}$$

$$= \frac{5 - 24 + 54 + 11}{4} = \frac{46}{4} = \frac{23}{2} = 11.5$$

Newton's Interpolation Formulae

① Newton's forward interpolation formula

Let the function $y = f(x)$ take the values y_0, y_1, y_2, \dots corresponding to the values $x_0, x_0+h, x_0+2h, \dots$ of x . Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$ where p is any real number.

$$E^p f(x) = f(x + ph)$$

$$y_p = f(x_0 + ph) = E^p f(x_0)$$

$$= (1 + \Delta)^p y_0$$

$$= \left[1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right] y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

+

② Newton's backward interpolation formula

Let the function $y = f(x)$ take the values y_0, y_1, y_2, \dots corresponding to the values $x_0, x_0+h, x_0+2h, \dots$ of x . Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$ where p is any real number.

$$y_p = f(x_n + ph) = E^p x_n = (1 - \nabla)^{-p} y_n$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!} \nabla^2 + \frac{p(p+1)(p+2)}{3!} \nabla^3 + \dots \right] y_n$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n$$

$$+ \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots$$

Numerical Integration

The process of evaluating the definite integral from a set of tabulated values of the integrand $f(x)$ is called numerical integration. i.e.,

$$\int_{x_0}^{x_0 + nh} f(x) dx$$

Quadrature formula

When the integrand is applied to a function of single variable is known as quadrature. This expression for the integral so obtained is called a quadrature formula.

Newton-Cotes's quadrature formula

$$\text{Let } I = \int_a^b f(x) dx$$

where $f(x)$ takes the values $y = y_0, y_1, y_2, \dots, y_n$ for $x = x_0, x_1, x_2, \dots, x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, \dots , $x_n = x_0 + nh$.

$$\text{Then } I = \int_{x_0}^{x_0 + nh} f(x) dx$$

$$= nh \left[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(n-3)}{12} \Delta^2 y_0 + \frac{n(n-1)^2}{24} \Delta^3 y_0 + \dots \right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Simpson's $\frac{1}{3}$ rd Rule

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

Q. Calculate by Simpson's rule an approximate value of $\int_{-3}^3 x^4 dx$ by taking seven equidistant ordinates. Compare it with the exact value and the value obtained by using the trapezoidal rule.

Sol.ⁿ

x	-3	-2	-1	0	1	2	3
$y = x^4$	81	16	1	0	1	16	81
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's Rule,

$$\int_{-3}^3 x^4 dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3} [(81 + 81) + 4(16 + 0 + 16) + 2(1 + 1)]$$

$$= \frac{1}{3} [162 + 128 + 4] = \frac{294}{3} = 98$$

The exact value

$$\int_{-3}^3 x^4 dx = \left[\frac{x^5}{5} \right]_{-3}^3 = \frac{3^5 - (-3)^5}{5} = \frac{243 + 243}{5} = 97.2$$

By Trapezoidal Rule

$$\int_{-3}^3 x^4 dx = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$
$$= \frac{1}{2} [(81 + 81) + 2(16 + 1 + 0 + 1 + 16)]$$
$$= 81 + 34 = 115$$

Q. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's

$\frac{1}{3}$ rule.

Solⁿ:

x	0	.2	.4	.6	.8	1
$y = \frac{1}{1+x^2}$	1					.5
	y_0	y_1	y_2	y_3	y_4	y_5

By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3} [(y_0 + y_5) + 4(y_1 + y_3) + 2(y_2 + y_4)]$$
$$= \frac{.2}{3} [$$

Q. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

i) Trapezoidal rule and

ii) Simpson's One-third rule.

Solⁿ

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$	0	.5	.2	.1			
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2} [$$

=

=

(approx.)

ii) By Simpson's One-third rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

=

=

=

(approx.)

Fourier Series

Note

$$(1) \int_x^{x+2\pi} \cos nx \, dx = 0$$

$$(2) \int_x^{x+2\pi} \sin nx \, dx = 0$$

$$(3) \int_x^{x+2\pi} \cos nx \cdot \cos nx \, dx = 0$$

$$(4) \int_x^{x+2\pi} \cos^2 nx \, dx = \pi$$

$$(5) \int_x^{x+2\pi} \sin nx \cdot \cos nx \, dx = 0$$

$$(6) \int_x^{x+2\pi} \sin nx \cdot \cos nx \, dx = 0$$

$$(7) \int_x^{x+2\pi} \sin nx \cdot \sin nx \, dx = 0$$

$$(8) \int_x^{x+2\pi} \sin^2 nx \, dx = \pi$$

$$\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$$

$$\cos\left(n + \frac{1}{2}\right)\pi = 0$$

$$\sin n\pi = 0$$

$$\cos n\pi = (-1)^n$$

Proof (1)
$$\int_x^{x+2\pi} \cos nx \, dx = \left[\frac{\sin nx}{n} \right]_x^{x+2\pi}$$

$$= \frac{\sin n(x+2\pi) - \sin nx}{n}$$

$$= \frac{2 \cos \frac{n(x+2\pi) + nx}{2} \cdot \sin \frac{n(x+2\pi) - nx}{2}}{n}$$

$$= \frac{2 \cos(n(x+\pi)) \cdot \sin n\pi}{n}$$

$$= \frac{2}{4} \cdot \cos \pi (\pi + \pi) = 0$$

$$= 0$$

$$(3) \int_{\kappa}^{\kappa+2\pi} \cos mx \cdot \cos nx \, dx = \frac{1}{2} \int_{\kappa}^{\kappa+2\pi} \{ \cos (m+n)x + \cos (m-n)x \} \, dx$$

$$= \frac{1}{2} \int_{\kappa}^{\kappa+2\pi} \{ \cos (m+n)x + \cos (m-n)x \} \, dx$$

$$= \frac{1}{2} \left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right]_{\kappa}^{\kappa+2\pi}$$

$$= \frac{1}{2} \left[\frac{\sin (m+n)(\kappa+2\pi) - \sin (m+n)\kappa}{m+n} + \frac{\sin (m-n)(\kappa+2\pi) - \sin (m-n)\kappa}{m-n} \right]$$

$$= \frac{1}{2} \left[\frac{2 \cos \frac{(m+n)(\kappa+2\pi) + (m+n)\kappa}{2} \cdot \sin \frac{(m+n)(\kappa+2\pi) - (m+n)\kappa}{2}}{m+n} \right.$$

$$\left. + \frac{2 \cos \frac{(m-n)(\kappa+2\pi) + (m-n)\kappa}{2} \cdot \sin \frac{(m-n)(\kappa+2\pi) - (m-n)\kappa}{2}}{m-n} \right]$$

$$= \frac{\cos \frac{(m+n)(\kappa+2\pi + \kappa)}{2} \sin \frac{(m+n)(\kappa+2\pi - \kappa)}{2}}{m+n}$$

$$+ \frac{\cos \frac{(m-n)(\kappa+2\pi + \kappa)}{2} \sin \frac{(m-n)(\kappa+2\pi - \kappa)}{2}}{m-n}$$

$$= \cos (m+n)(\kappa + \pi) \cdot \sin (m+n)\pi / (m+n) + \cos (m-n)(\kappa + \pi) \cdot \sin (m-n)\pi / (m-n)$$

$$= \frac{0}{m+n} + \frac{0}{m-n} = 0$$

Euler's Formulae

The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

These values of a_0, a_n, b_n are known as Euler's formulae.

Proof

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

To find a_0

integrate both sides of (1) from $x = \alpha$ to

$$x = \alpha + 2\pi,$$

$$\int_{\alpha}^{\alpha+2\pi} f(x) dx = \int_{\alpha}^{\alpha+2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \left[\frac{a_0}{2} x + \sum_{n=1}^{\infty} a_n \cdot \frac{\sin nx}{n} + \sum_{n=1}^{\infty} b_n \cdot \frac{-\cos nx}{n} \right]_{\alpha}^{\alpha+2\pi}$$

$$= \frac{a_0}{2} (\alpha + 2\pi - \alpha) + \sum_{n=1}^{\infty} \frac{a_n}{n} \left\{ \sin n(\alpha + 2\pi) - \sin n\alpha \right\} - \sum_{n=1}^{\infty} \frac{b_n}{n} \left\{ \cos n(\alpha + 2\pi) - \cos n\alpha \right\}$$

$$\begin{aligned}
&= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx dx \\
&= \frac{a_0}{2} [x]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty} a_n \cdot 0 + \sum_{n=1}^{\infty} b_n \cdot 0 \\
&= \frac{a_0}{2} (\alpha + 2\pi - \alpha) + 0 + 0 \\
&= \frac{a_0}{2} \cdot 2\pi = a_0 \pi \\
\Rightarrow a_0 &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx
\end{aligned}$$

To find a_n

Multiplying each sides of (1) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx = \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \cos nx dx + \int_{\alpha}^{\alpha+2\pi} \sum_{n=1}^{\infty} a_n \cos^2 nx dx$$

$$\begin{aligned}
&+ \int_{\alpha}^{\alpha+2\pi} \sum_{n=1}^{\infty} b_n \sin nx \cos nx dx \\
&= \frac{a_0}{2} \cdot 0 + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos^2 nx dx \\
&+ \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \cdot \cos nx dx
\end{aligned}$$

$$= 0 + a_n \cdot \pi + b_n \cdot 0$$

$$= a_n \pi$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

To find b_n

Multiplying each sides of ① by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$,

$$\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx = \int_{\alpha}^{\alpha+2\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right) \cdot \sin nx \, dx$$

$$= \frac{a_0}{2} \int_{\alpha}^{\alpha+2\pi} \sin nx \, dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \cdot \sin nx \, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx$$
$$= \frac{a_0}{2} \cdot 0 + \sum_{n=1}^{\infty} a_n \cdot 0 + \sum_{n=1}^{\infty} b_n \cdot \pi$$

$$= 0 + 0 + b_n \pi$$

$$= b_n \pi$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx$$

Proved