classmate
EXPRESSIONS

Control System
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Mt. EVEREST

WORLD'S HIGHEST MOUNTAIN PEAK

Notebook

and applications"
nverters

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		Subhasis Nanda.		

Basic Concept of Control System.

Definitions :-

Controlled Variable :- It is the quantity or condition that is measured and controlled. Generally it is the output of the system.

Manipulated Variable :- It is the quantity or condition that is ~~meas~~ varied by the controller so as to affect the value of the controlled variable.

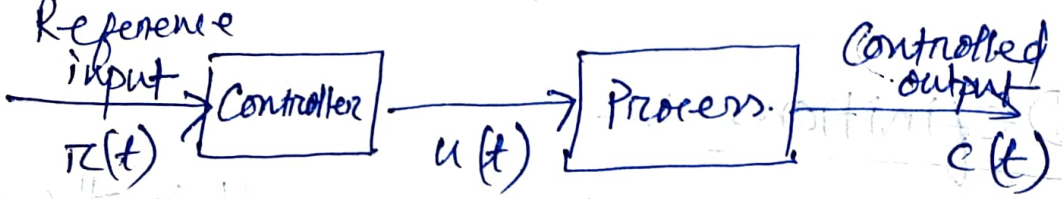
Plant :- A plant may be a piece of equipment, a set of machine parts functioning together, the purpose of which is to perform a particular operation.

System :- A system is a combination of components that act together and perform a certain objective.

Control :- The meaning of control is to regulate, direct or command a system so that a desired objective is obtained.

Control System :- It is a combination of different physical ~~system~~ elements linked together in such a manner ~~that~~ so as to regulate, direct or command itself to obtain a certain objective.

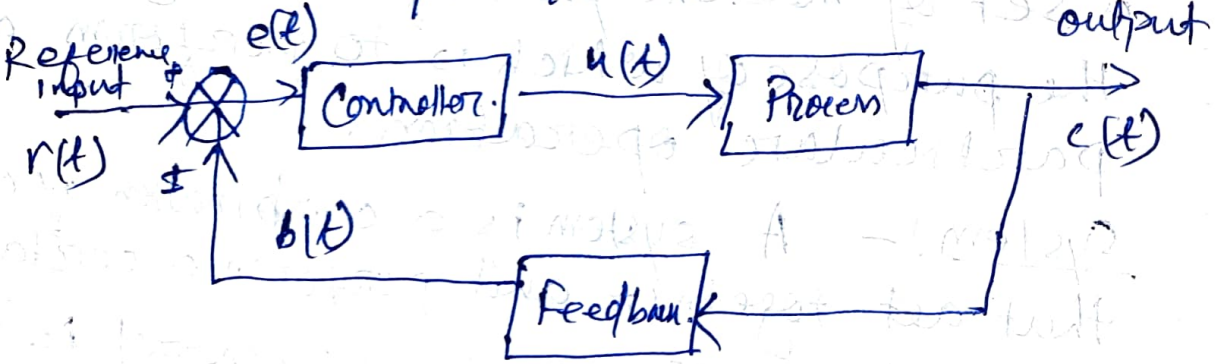
Open Loop system :-



A system in which control action does not depends on the output is known as open-loop system.

Example - Fan with Regulator,
Semi-automatic washing machine.
bread toaster, coffee server.

Closed loop System :-



If the controlling action of a system are somehow dependent on output or changes in output, system is called as closed loop system.

Here r(t) = reference Input
e(t) = Error signal.
u(t) = Actuating signal
c(t) = controlled output

$$e(t) = r(t) \pm b(t)$$

The $e(t)$ is applied to the controller, which produces the acting signal $u(t)$ for the process in such a manner that $e(t)$ will be reduced to zero.

Example :- Automatic Electric Iron,
Servo voltage stabilizer.
Water level controller.

Open loop Control system	Closed loop control system.
1. The control action does not depend upon the output of the system.	1. The control action depends on the output of the system.
2. There is no feedback element.	2. There is feedback element.
3. It is a stable as the output is just a function of input and control parameter.	3. It is less stable as any change in output changes the input and hence the output changes again.
4. Sensitive to system parameter.	4. Less sensitive to system parameter.
5. It is simple and easy to build. It is cheap.	5. It is complicated and cost is more.
6. Fewer components required.	6. More components require
7. Example - Fan with Regulat	7. Exam- Automatic Electric Ir

Transfer function:-

Basically the transfer function of a system means the ratio of output variable to the input variable.

But in most of our physical system, the input and output variables are in totally different units and hence we unable to clearly state the ratio. To overcome this the variables are converted to their corresponding Laplace transform and the ratio is said to be their Transfer function.

⇒ Transfer function of a control system is defined as the ratio of Laplace transformation of the output variable to the Laplace transformation of the input variable assuming initial condition as zero.

$$T(s) = \left.\frac{C(s)}{R(s)}\right|_{t=0,\, c(t)=0}$$

Mathematical Models of Physical systems.

Physical system!— It is a collection of physical objects connected together to serve an objective.

→ Physical System cannot be represented in its full physical details and therefore assumptions are always made for analysis of systems.

→ An idealised physical system is called as a "Physical Model".

→ A mathematical model is the mathematical representation of the physical Model through use of appropriate physical laws.

→ A control system may be modelled as a scalar differential equation describing the system or state variable vector-matrix differential equation.

→ Physical system are represented as interconnection of idealised system elements and described in forms of elemental laws.

There are two basic assumption!—

1. Mass which has physical dimentions, is considered concentrated at a point and temperature in a room is replaced by a representive temprature at that point.
2. We assume that the variable associated with the elements lie in the range that the element can be described by simple linear law of i) a constant of proportionality

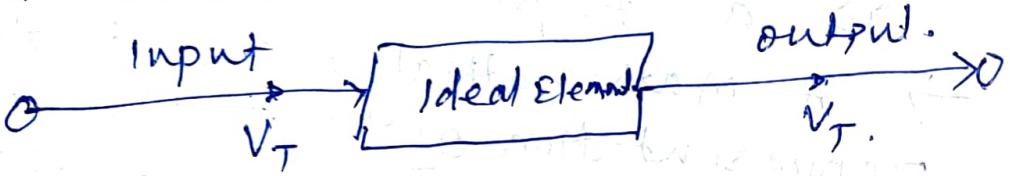
to x

ii) a first order derivative.
iii) a first order integration.

Let us assume a ideal element which have a single point, two terminal representation. So the variables associated with it are.

i) Through variable (V_T) -

It passes through the element and has same value at input and output.

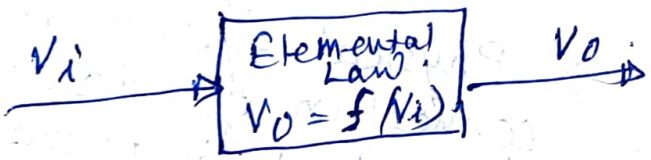


ii) Across variable (V_A)

It appears across the element.

Another classification of variable elements.

i) Input variable ii) Output variable.



Mechanical Systems:

These can be modelled by means of three ideal translatory and three ideal rotary elements.

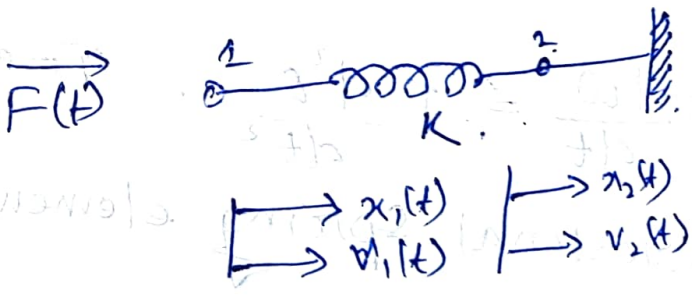
Translatory Models:-

1. The mass element.

M = Mass of Element
x = displacement
v = velocity.
F = applied force

$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$

2. The spring element :-

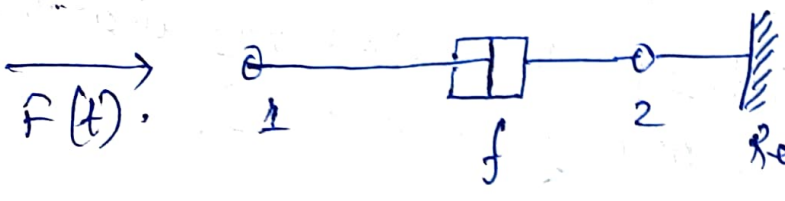


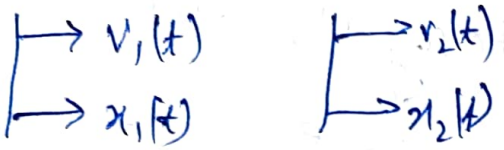
K = spring constant
x = net compression
v = observed velocity.

$$F = K(x_1 - x_2) = Kx$$

$$= K\int_{-\infty}^{t}(v_1 - v_2)dt = K\int_{-\infty}^{t} v\,dt$$

3. The damper element.



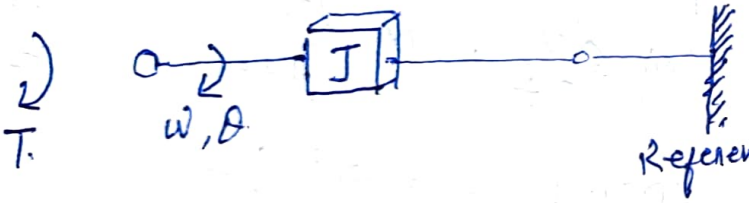


f = damper Constant.

$$F = f(v_1 - v_2) = fv = f\left[\frac{dx_1}{dt} - \frac{dx_2}{dt}\right] = f\frac{dx}{dt}.$$

Rotational Model.

1. The Inertia Element.

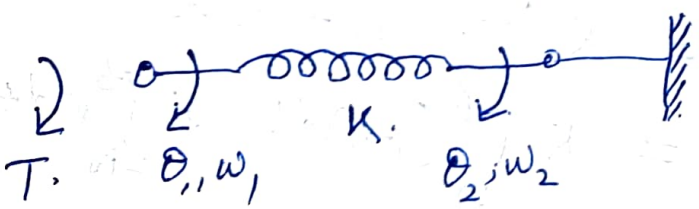


θ = angular displacement
w = angular velocity
J = Moment of Inertia

$$T = J\frac{dw}{dt} = J\frac{d^2\theta}{dt^2}$$

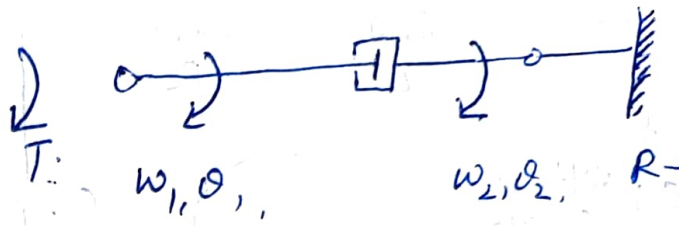
T = applied Torque

2. The torsional spring element.



$$T = K(\theta_1 - \theta_2) = K\theta$$
$$= K\int_{-\infty}^{t}(w_1 - w_2)\,dt = K\int_{-\infty}^{t} w\,dt.$$

3. The damper element.

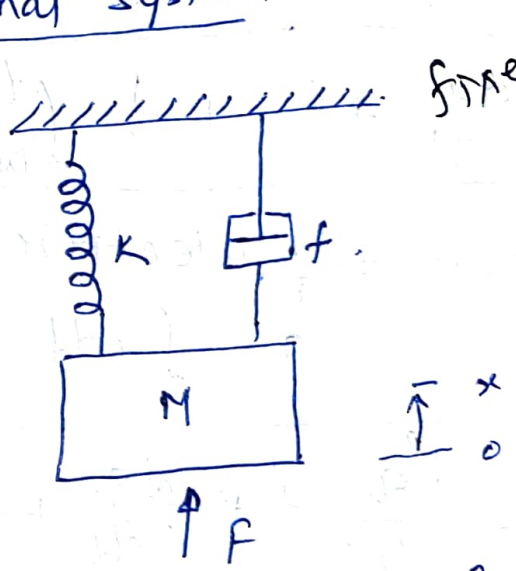


$$T = f(\omega_1 - \omega_2) = f\omega$$

$$= f\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = f\left(\frac{d\theta}{dt}\right)$$

— x —

Translational System



$$F - f\frac{dx}{dt} - Kx = M\frac{d^2x}{dt^2}$$

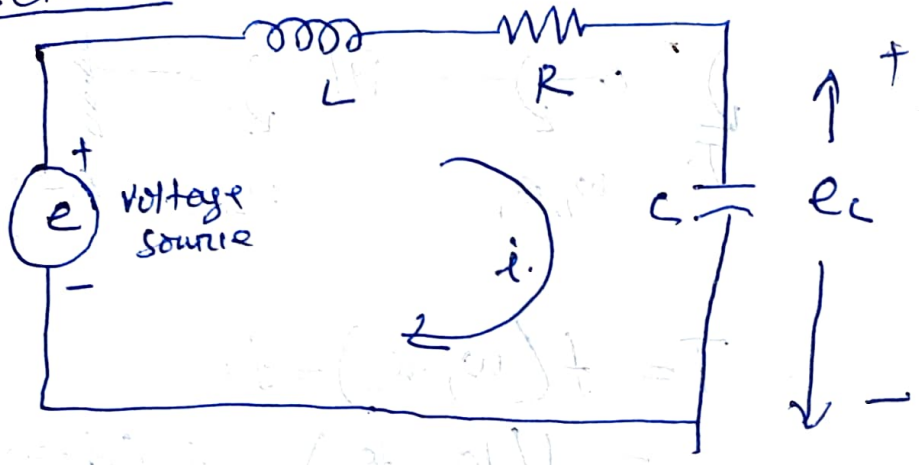
$$\Rightarrow F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx \quad ---(1)$$

For a rotational system.

$$T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K\theta \quad ---(2)$$

Electrical Systems

In series



By using Kirchoff's voltage law.

The equation of the system are.

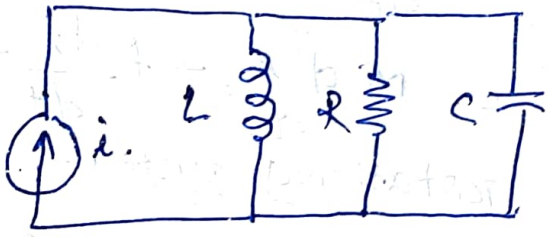
$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{0}^{t} i\,dt = e$$

In terms of electric charge

$$q = \int i\,dt \Rightarrow i = \frac{dq}{dt}$$

$$\therefore \quad L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e$$

In Parallel



$$C\frac{de}{dt} + \frac{1}{L}\int_{-\infty}^{t} e\,dt + \frac{e}{R} = i.$$

In terms of magnetic flux linkage.

$$\phi = \int e\,dt. \Rightarrow e = \frac{d\phi}{dt}.$$

$$\therefore \quad C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi = i$$

Analogous Systems

① Analogous Quantities in Force/Torque – Voltage

Translational System	Rotational system.	Electrical system.
Force (F)	Torque (T)	Voltage (e)
Mass (M)	Moment of Inertia (J)	Inductance (L)
Viscous friction (f)	Viscous friction (f)	Resistance (R)
Spring constant (K)	Spring constant (K)	Inverse of capacitance (1/c)
Displacement (x)	Angular Displacement (θ)	charge (q)
velocity (v)	Angular velocity (ω)	Current (i)

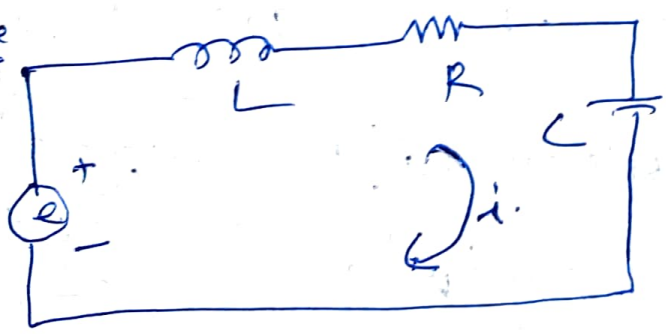
Analogous Quantity in Force / Torque - Current -

Translational System.	Rotational system.	Electrical system.
F	T	i.
M	J	C
f	f	1/R
K	K	1/L
x	θ	∅
v	ω	e.

— X —

Derivation of Transfer function:-

Example



Sol

KVL is.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = e$$

Taking Laplace transformation of both side.

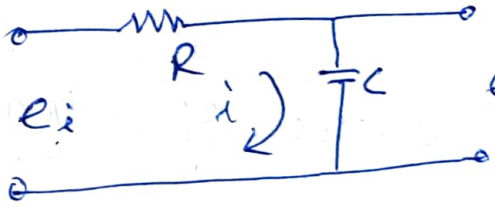
$$L\,sI(s) + RI(s) + \frac{I(s)}{Cs} = E(s).$$

Taking e as the input parameter and i as the output parameter.

$$\frac{I(s)}{E(s)} = \frac{I(s)}{I(s)\left[Ls + R + \frac{1}{Cs}\right]}$$

$$\boxed{\frac{I(s)}{E(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}}$$ Ans.

Example



Sol.

$$e_i = Ri + \frac{1}{C}\int i\,dt$$

T. L. T.

$$E_i(s) = RI(s) + \frac{1}{Cs}I(s) = I(s)\left(R + \frac{1}{Cs}\right)$$

&

$$e_0 = \frac{1}{C}\int i\,dt$$

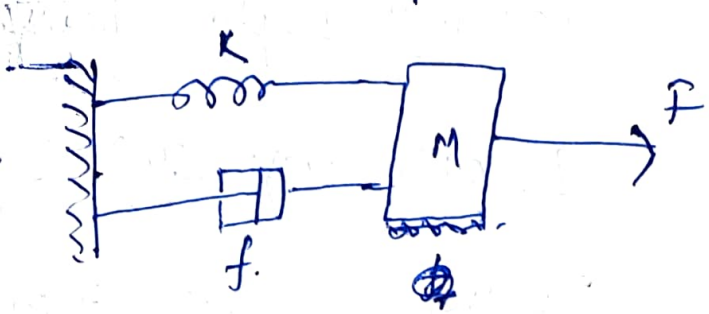
T. L. T

$$E_0(s) = \frac{1}{Cs}I(s)$$

∴ Transfer function is given by.

$$\frac{E_0(s)}{I_0(s)} = \frac{\frac{1}{Cs}I(s)}{\left(R + \frac{1}{Cs}\right)I(s)} = \frac{1}{1 + RCs}$$ Ans

Example



$$F = Kx + f\frac{dx}{dt} + M\frac{d^2x}{dt^2}$$

$$\Rightarrow F(s) = K X(s) + fs X(s) + Ms^2 X(s)$$

$$= X(s)\left[K + fs + Ms^2\right].$$

$$\therefore \frac{X(s)}{F(s)} = \frac{1}{K + fs + Ms^2}$$ Ans.

Poles and Zeros of a transfer function:-

The transfer function of a control system can be represented in the form of a quotient of polynomials in the following form:

$$G(s) = \frac{A(s)}{B(s)} = \frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} \cdots + a_n}{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_m}$$

Further the numerator and the denominator can be factored into n and m terms respectively.

$$G(s) = \frac{A(s)}{B(s)} = \frac{K(s-s_1)(s-s_2)\cdots(s-s_n)}{(s-s_a)(s-s_b)\cdots(s-s_m)}$$

(Poles & Zeros representation of Transfer fund)

Where $K = \frac{a_0}{b_0}$ is known as the gain factor of the transfer function.

If the numerator is equated to zero, then n roots of the equation are $s_1, s_2, s_3 \cdots s_n$ and equating the denominator to zero, the, the m roots can be determined as.

$s_a, s_b, s_c \cdots s_m$.

Poles - The roots of the denominator are called as poles of transfer function as if we put the value of s equal to these roots the transfer function becomes infinity.

Zeros - The roots of the numerator ~~is equated to zero~~ are called the zeros of transfer function, as if we put the value of s to these values then the transfer function becomes zero.

The Poles & zeros representation of transfer function can be re-arranged as.

$$G(s) = \frac{A(s)}{B(s)} = \frac{K'(1-sT_1)(1-sT_2)\cdots(1-sT_n)}{(1-sT_a)(1-sT_b)\cdots(1-sT_m)}$$

where $K' = \frac{K \cdot s_1 \cdot s_2 \cdots s_n}{s_a \cdot s_b \cdots s_m}$ and $T_1 = 1/s_1, T_2 = 1/s_2$ $T_a = 1/s_a, T_b = 1/s_b$ etc.

→ This form of representation is called as Time constant form of transfer function

→ $T_1, T_2 \cdots$ and $T_a \cdots T_b$ are the time constantants of the individual blocks or elements of the control system.

"Order" and "Type" of Control System.

Order - The highest power of the denominator polynomial of a transfer function is called as the order of control system.
(m is the order of CS)

Suppose in the poles zeros representation of the transfer function, some of the zeros and poles are zero i.e $(s_1 = 0, s_5 = 0, s_b = 0, s_4 = 0)$

we can rewrite the transfer function as

$$G(s) = \frac{K(s - s_1')(s - s_2') \cdots}{s^N (s - s_a')(s - s_b') \cdots}$$

Type - It is the number of poles at origin of a control system.
N - is the type of CS

Sensitivity

$$\text{Sensitivity} = \frac{\%\text{ change in } A}{\%\text{ change in } K}.$$

Where as K is the system parameter such as gain or feed back and A is the response of the system such as output.

Mathematically denoted by.

$$S_K^A = \frac{\partial A/A}{\partial K/K} = \frac{\partial A}{\partial K} \cdot \frac{K}{A}.$$

i) Sensitivity of overall transfer function $M(s)$ with respect to forward path transfer function $G(s)$.

It is denoted by S_G^M.

$$S_G^M = \frac{\partial M(s)}{\partial G(s)} \cdot \frac{G(s)}{M(s)}.$$

For open loop control system:-

$$M(s) = G(s) \Rightarrow \frac{G(s)}{M(s)} = 1.$$

$$\frac{\partial M(s)}{\partial G(s)} = 1 \quad \therefore \quad S_G^M = 1.$$

For closed loop control system.

$$M(s) = \frac{G(s)}{1 + G(s)H(s)}.$$

$$\frac{G(s)}{M(s)} = 1 + G(s)H(s).$$

$$\therefore \frac{dM(s)}{dG(s)} = \frac{1}{1+G(s)H(s)} - \frac{G(s)H(s)}{\{1+G(s)H(s)\}^2}$$

$$= \frac{1 + G(s)H(s) - G(s)H(s)}{\{1+G(s)H(s)\}^2}$$

$$= \frac{1}{\{1+G(s)H(s)\}^2}.$$

$$\therefore S_G^M = \frac{1+G(s)H(s)}{\{1+G(s)H(s)\}^2} = \frac{1}{1+G(s)H(s)}.$$

$$\Rightarrow \boxed{S_G^M = \frac{1}{1+G(s)H(s)}}.$$

ii) Sensitivity of overall transfer function $M(s)$ with respect to feedback path transfer function $H(s)$.

$$S_H^M = \frac{dM}{dH} \cdot \frac{H}{M}. \qquad M(s) = \frac{G_1(s)}{1 + G_1(s) H(s)}.$$

$$\frac{dM(s)}{dH(s)} = \frac{-\{G_1(s)\}^2}{\{1 + G_1(s) H(s)\}^2}.$$

$$\frac{H(s)}{M(s)} = \frac{H(s)}{G_1(s) / 1 + G_1(s) H(s)} = \frac{H(s)\{1 + G_1(s) H(s)\}}{G_1(s)}.$$

$$\therefore S_H^M = \frac{-\{G_1(s)\}^2}{\{1 + G_1(s) H(s)\}^2} \cdot \frac{H(s)\{1 + G_1(s) H(s)\}}{G_1(s)}.$$

$$\boxed{S_H^M = \frac{-G_1(s) H(s)}{1 + G_1(s) H(s)}}$$

$$\begin{cases} f(x) = \dfrac{A}{1 + Ax}. \\ f'(x) = \end{cases}$$

Effect of feedback on time constant of a control system

Considering a open loop control system with transfer function

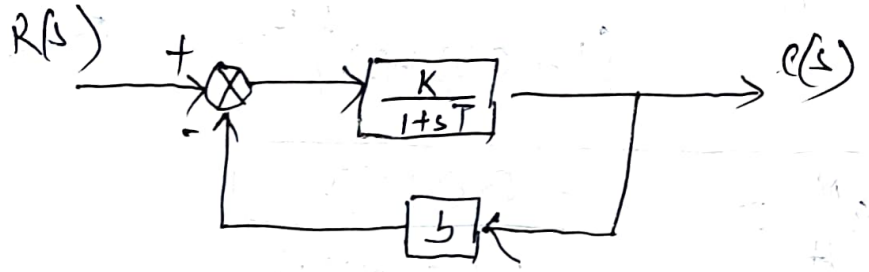
$$\frac{C(s)}{R(s)} = \frac{K}{1+sT}$$

, K is the gain.
T is the time constant

The response of the system to unit step input is

$$c(t) = \frac{K}{T}\left(1 - e^{-t/T}\right)$$

Now a feedback element is added with feedback path gain b.



$$\therefore \text{now } \frac{C(s)}{R(s)} = \frac{K}{1+sT+Kb} = \frac{K}{1+Kb}\left[\frac{1}{1+s\left(\frac{T}{1+Kb}\right)}\right]$$

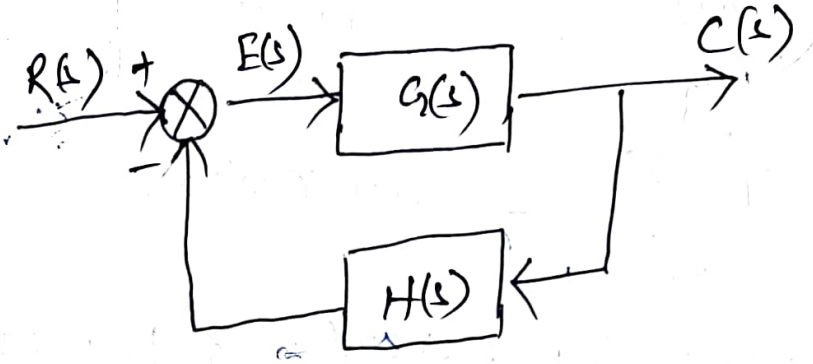
So the time constant changes to

$$T/1+Kb$$

as $K>1$ & b is a +ve , $\frac{T}{1+Kb} < T$

That means the ~~[illegible]~~ decreases

Steady State Error



The product $G(s)H(s)$ is called as open loop transfer function.

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

$$\Rightarrow E(s) = R(s) \cdot \frac{1}{1+G(s)H(s)}.$$

Steady state error can be determined at $t \to \infty$, and applying final value theorem.

$$e_{ss} = \lim_{s \to 0} s\, E(s) = \lim_{s \to 0} s\, R(s) \frac{1}{1+G(s)H(s)}.$$

— x —

The output of a control system is called as "position" in any physical form.

The first derivative of output is called "velocity" and second derivative is called as "acceleration".

$\therefore$ $c(t)$ → position.

$\frac{dc(t)}{dt}$ → velocity.

$\frac{d^2c(t)}{dt^2}$ → acceleration.

Static Error coefficient → { for unit step input }.

i) Positional error coefficient. (K_p).

K_p can be determined as.

$$e_{ss} = \lim_{s \to 0} s \, R(s) \frac{1}{1 + G(s)H(s)}.$$

$$R(s) = \frac{1}{s}.$$

$$\therefore e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s} \frac{1}{1 + G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)H(s)}.$$

$$\lim_{s \to 0} G(s)H(s) = K_p$$

$$\therefore e_{ss} = \frac{1}{1 + K_p}.$$

ii) Static velocity error coefficient (K_v)

K_v is associated with unit ramp input applied to a closed-loop control system.

$$e_{ss} = \lim_{s \to 0} s \, R(s) \frac{1}{1 + G(s)H(s)}.$$

$$R(s) = \frac{1}{s^2}$$

$$\therefore e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s^2} \frac{1}{1 + G(s)H(s)}.$$

$$= \lim_{s \to 0} \frac{1}{s + s\,G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s\,G(s)H(s)} = \frac{1}{\lim_{s \to 0} s\,G(s)H(s)}.$$

$$\lim_{s \to 0} s\,G(s)H(s) = K_v$$

$$\therefore e_{ss} = \frac{1}{K_v}.$$

iii) Static acceleration error coefficient (K_a)

K_a is associated with unit parabolic input.

$$e_{ss} = \lim_{s \to 0} s \, R(s) \frac{1}{1 + G(s) H(s)}.$$

$$R(s) = \frac{1}{s^3}$$

$$\therefore e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s) H(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s) H(s)}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s) H(s)}.$$

$$\lim_{s \to 0} s^2 G(s) H(s) = K_a.$$

$$\therefore e_{ss} = \frac{1}{K_a}.$$

Types of transfer function & e_{ss}

e_{ss} depend upon the type of open-loop transfer function of a closed-loop control system.

1. a) Type 0 system ~~for positional input~~ with unit step input.

$$G(s) H(s) = \frac{K(1 + sT_a)(1 + sT_b) \cdots}{(1 + sT_1)(1 + sT_2) \cdots}$$

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$$\Rightarrow K_p = K.$$

$$\boxed{e_{ss} = \frac{1}{1+K}}$$

1.b Type 0 system with unit ramp input.

$$K_v = \lim_{s \to 0} s \frac{K(1+sT_a)(1+sT_b)\cdots}{(1+sT_1)(1+sT_2)\cdots}$$

$$= 0$$

$$\therefore \boxed{e_{ss} = \frac{1}{K_v} = \infty}$$

1.3 Type 0 system with unit parabolic input

Similarly $K_v = 0$.

$$\therefore \boxed{e_{ss} = \infty}$$

2. Type '1' system.

a) Unit step input.

$$G(s)H(s) = \frac{K(1+sT_a)(1+sT_b)\cdots}{s(1+sT_1)(1+sT_b)\cdots}$$

$$\therefore K_p = \lim_{s \to 0} \frac{K(1+sT_a)(1+sT_b)}{s(1+sT_1)(1+sT_b)\cdots} = \infty.$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0.$$

$$\Rightarrow \boxed{e_{ss} = 0}$$

b) for unit ramp input.

$$K_v = \lim_{s \to 0} s \frac{K(1+sT_a)(1+sT_b)\cdots}{s(1+sT_a)(1+sT_2)\cdots}$$

$$= K.$$

$$\therefore \boxed{e_{ss} = 1/K_v}$$

c) for unit parabolic input

$K_a = 0$.

$$\boxed{e_{ss} = \infty}$$

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Input	L(input)	type '0'		type '1'		type '2'	
		Static error coeff	e_{ss}	Static error coeff	e_{ss}	Static error coeff	e_{ss}
Unit step	$\frac{1}{s}$	$K_p = K$	$\frac{1}{1+K}$	$K_p = \infty$	0	$K_p = \infty$	0
Unit ramp	$\frac{1}{s^2}$	$K_v = 0$	∞	$K_v = K$	$\frac{1}{K}$	$K_v = \infty$	0
Unit Parabolic	$\frac{1}{s^3}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = K$	$\frac{1}{K}$

Generalised Error co-efficient

Steady state error determined by. state error co-efficient is calculated for step, ramp and parabolic input only.

Generalised error co-efficient provides steady state error performance of a feedback control system to any arbitrary input function. Here the error is indicated in terms of time and final steady state error is determined as t approaches ∞.

Let the error transfer function of a feedback control system is given by.

$$F_e(s) = \frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}.$$

$$\therefore \quad E(s) = f_e(s) \cdot R(s)$$

Using convolution theorem & applying inverse laplace transform

$$e(t) = f_e(t) * r(t-t)$$

$$= \int_{-\infty}^{t} f_e(\tau) \cdot r(t-\tau)\, d\tau .$$

the function $r(t-\tau)$ is valid for $t \geq 0$.

so.

$$e(t) = \int_{0}^{t} f_e(\tau) \cdot r(t-\tau)\, d\tau .$$

By using taylor series expansion

$$r(t-\tau) = r(t) - \tau r'(t) + \frac{\tau^2}{2!} r'(t) - \frac{\tau^3}{3!} r'''(t) - \cdots$$

$$\therefore e(t) = \int_{0}^{t} f_e(\tau) \left\{ r(t) - \tau r'(t) + \frac{\tau^2}{2!} r'(t) \cdots \right\} d$$

steady state error is expressed at $t \to \infty$.

so $$\lim_{t \to \infty} e(t) = e_{ss} = \lim_{t \to \infty} \int_{0}^{t} f_e(\tau) \left\{ r(t) - \tau r'(t) + \frac{\tau^2}{2!} r'(t) - \cdots \right\} d\tau$$

$$= \int_0^\infty r(t) f_e(z)dz - \int_0^\infty r'(t) z f_e(z)dz + \int_0^\infty r''(t)\frac{z^2}{2!} f_e(z)dz - \cdots$$

$$= r(t)\int_0^\infty f_e(z)dz - r'(t)\int_0^\infty z f_e(z)dz + r''(t)\int_0^\infty \frac{z^2}{2!} f_e(z)dz - \cdots$$

Now Put

$$C_0 = \int_0^\infty f_e(z)dz.$$

$$C_1 = -\int_0^\infty z f_e(z)dz.$$

$$C_2 = \int_0^\infty \frac{z^2}{2!} f_e(z)dz$$

$$C_n = (-)^n \int_0^\infty \frac{z^n}{n!} f_e(z)dz.$$

$$\therefore \boxed{e_{ss} = C_0 r(t) + C_1 r'(t) + \frac{C_2}{2!} r''(t) + \frac{C_3}{3!} r'''(t) - \cdots + \frac{C_n}{n!} r^n(t).}$$

The coefficient C_0, C_1, C_2 are termed as generalised error coefficient the Expression is known as generalised error series.

as $F_e(s) = \int_0^\infty f_e(z)\, e^{-sz}\, dz$.

$$\therefore \lim_{s\to 0} F_e(s) = \lim_{s\to 0} \int_0^\infty f_e(z)\, e^{-sz}\, dz$$

$$= \int_0^\infty f_e(z)\, dz = C_0.$$

$$\therefore C_0 = \lim_{s\to 0} F_e(s).$$

$$\text{and } \frac{d\,F_e(s)}{ds} = (-)\int_0^\infty z f_e(z)\, e^{-sz}\, dz.$$

$$\lim_{s\to 0} \frac{d\,F_e(s)}{ds} = (-1)\int_0^\infty z\, f_e(z) e^{-sz}\, dz$$

$$= (-)\int_0^\infty z\, f_e(z)\, dz = C.$$

$$\therefore C_1 = \lim_{s\to 0} \frac{d\,F_e(s)}{ds}$$

Similarly

$$C_2 = \lim_{s\to 0} \frac{d^2 F_e(s)}{ds^2}$$

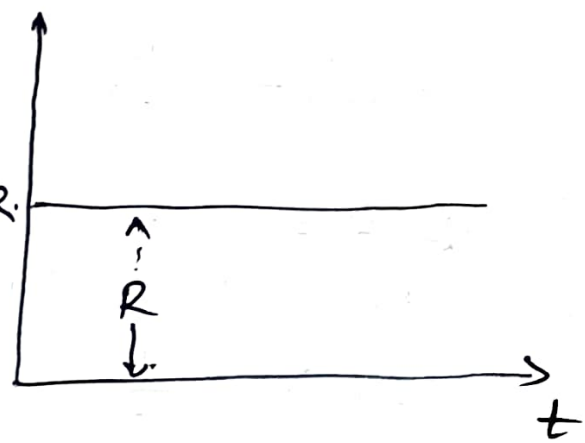
$$\boxed{C_n = \lim_{s\to 0} \frac{d^n F_e(s)}{ds^n}.}$$

Time Response Anayesis of Control System.

Standard test signal

i) Step Input signal

It is an input of constant value.



$r(t) = R$ for $t \geq 0$

$= 0$ for $t \leq 0$

If $R = 1$,

$r(t) = 1$

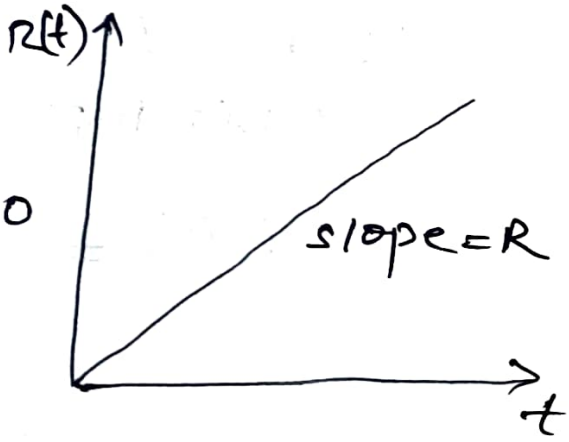
and the step input is called as a unit step input.

$$R(s) = L\{r(t)\} = L\{R\} = R/s$$

for unit step input

$$R(s) = 1/s$$

ii) Ramp input signal



$r(t) = R.t$ for $t \geq 0$

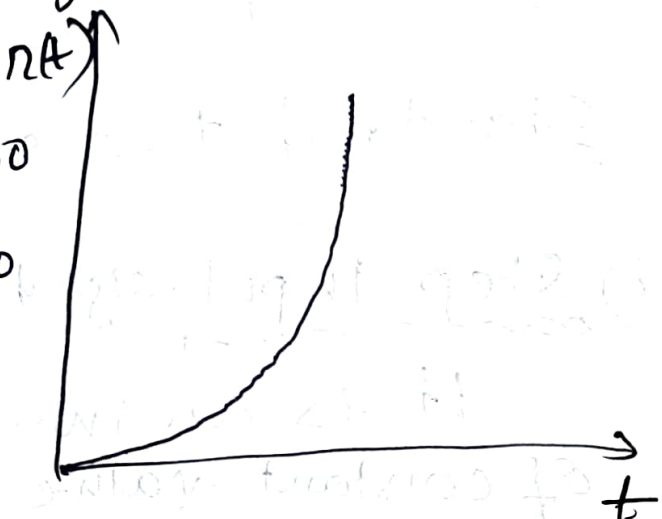
$= 0$ $t < 0$

$$R(s) = R/s^2$$

If $R = 1$, it is called as unit ramp input.

$r(t) = t$, $R(s) = \frac{1}{s^2}$

(iii) Parabolic Input signal.



$$r(t) = \frac{Rt^2}{2} \quad \text{for } t \geq 0$$

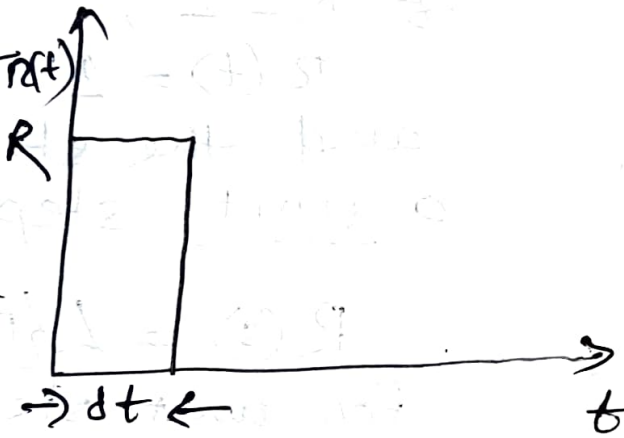
$$= 0 \quad \text{for } t < 0$$

$$R(s) = \frac{R}{s^3}$$

If $R = 1$.
then it is called as unit parabolic input

$$R(s) = \frac{1}{s^3}$$

iv) Impulse input signal



$$r(t) = R \quad 0 \leq t \leq dt$$

$$= 0 \quad \text{otherwise}$$

whereas $dt \cong 0$

$R(s) = R$.

If $R = 1$, then it is called unit impulse signal.

$$R(s) = 1.$$

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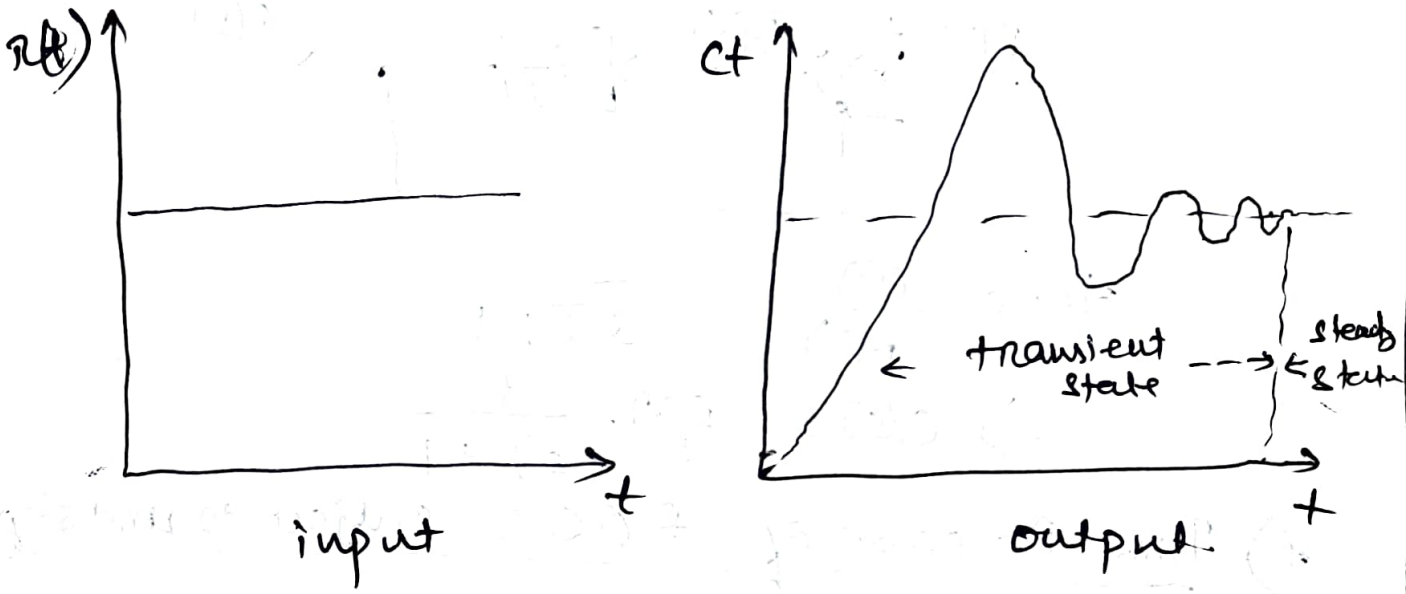
Transient Response and Steady State Response



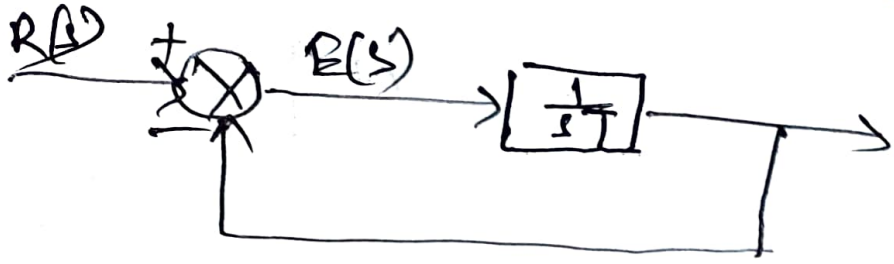
→ If we applied an step input to a control system, Generally the initial part of the output of control system is transient in nature, or in other words the output fluctuates or unsteale.

→ After a certain time period (t_s) the output stabilised and is called as steady state response.

→ Theoritically a steady state means a state of the output of a control system as the time approaches infinity after initiation of the input.

→ Practically, the output said to reach steady state value when the actual output reaches within 2% of the desired value. Generally it is identified in terms of Time constant of the control system.

Response of First order Control System.



Here $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

$\Rightarrow C(s) = R(s) \cdot \frac{1}{sT+1}$

i) Time Response of FOCS subject to unit step input

for unit step input

$r(t) = 1$

$R(s) = 1/s$.

$\therefore C(s) = \frac{1}{s} \cdot \frac{1}{sT+1}$.

applying partial fraction on R.H.S.

$C(s) = \frac{1}{s} - \frac{1}{s+1/T}$

Taking inverse laplace transformation

$$\boxed{c(t) = 1 - e^{-t/T}}$$

As we know $e(t) = R(t) - C(t)$

$\therefore e(t) = 1 - 1 + e^{-t/T} = e^{-t/T}$

$$\boxed{e(t) = e^{-t/T}}$$

at steady state the error which is equal to zero is called as steady state error.

$$\therefore \lim_{t \to \infty} e(t) = 0$$

$$\Rightarrow \lim_{t \to \infty} e^{-t/T} = 0.$$

At steady state, the error is called as steady state Error and represented as E_{ss}.

$$E_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} e^{-t/T} = 0$$

for $t = T$, $e(t) \simeq 0.368$

$t = 4T$, $e(t) \simeq 0.02$

$t = \infty$, $e(t) \simeq 0$.

→ So $4T$ is the basic demarcation between transient part and steady state part in the analysis of the output of a control system.

$t_s = 4T$ is called as settling time of control system.

→ If we decrease the time constant of a control system, then the steady state will reaches faster.

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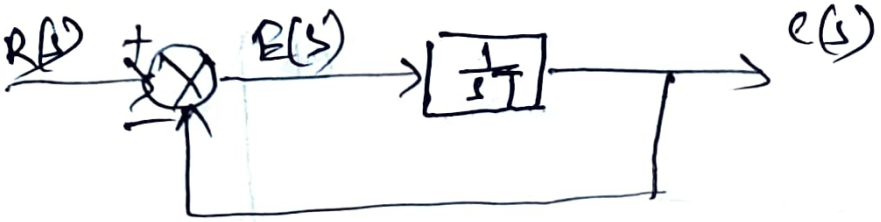
Time Response of First order Control System



Here $\frac{C(s)}{R(s)} = \frac{1}{sT+1}$

$\Rightarrow C(s) = R(s) \cdot \frac{1}{sT+1}$

i) Time Response of FOCS subject to unit step input

for unit step input

$r(t) = 1$

$R(s) = 1/s$.

$\therefore C(s) = \frac{1}{s} \cdot \frac{1}{sT+1}$.

applying partial fraction on R.H.S.

$C(s) = \frac{1}{s} - \frac{1}{s+1/T}$

Taking inverse laplace transformation

$$\boxed{c(t) = 1 - e^{-t/T}}$$

As we know $e(t) = R(t) - C(t)$

$\therefore e(t) = 1 - 1 + e^{-t/T} = e^{-t/T}$

$$\boxed{e(t) = e^{-t/T}}$$

at steady state the error must be equal to zero it is called as steady state error

$$\therefore \lim_{t \to \infty} e(t) = 0$$

$$\Rightarrow \lim_{t \to \infty} e^{-t/T} = 0$$

At steady state, the error is called as steady state Error and represented as E_{ss}.

$$E_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} e^{-t/T} = 0$$

for $t = T$, $e(t) \cong 0.368$

$t = 4T$, $e(t) \cong 0.02$

$t = \infty$, $e(t) \cong 0$.

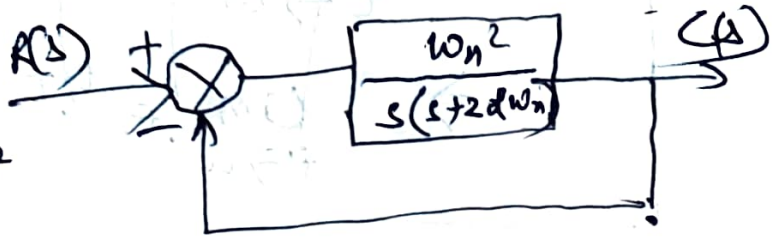
→ So $4T$ is the baste demarcation between transient part and steady state part in the analysis of the output of a control system.

$t_s = 4T$ is called as settling time of control system.

→ If we decrease the time constant of a control system, then the steady state will reaches faster.

—X—

Time Response of Second order Control system subject to unit step input :—



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\therefore C(s) = R(s) \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\therefore r(t) = 1 \quad , \quad R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

by partial fraction.

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

by rearranging.

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

Put $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

Taking inv. Laplace transformation,

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$+\left[1+\frac{\zeta\omega_n}{\omega_d}\sqrt{1-\zeta^2}\cos\omega_d t\right]$$

as $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t\right]$$

Put $\zeta = \cos\phi$, $\sqrt{1-\zeta^2} = \sin\phi$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\phi\cos\omega_d t - \cos\phi\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin[\omega_d t + \phi]$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]}$$

$$e(t) = 1 - c(t)$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

— X —

$\omega_n \rightarrow$ Natural frequency of oscillation

$\zeta \rightarrow$ Damping ratio.

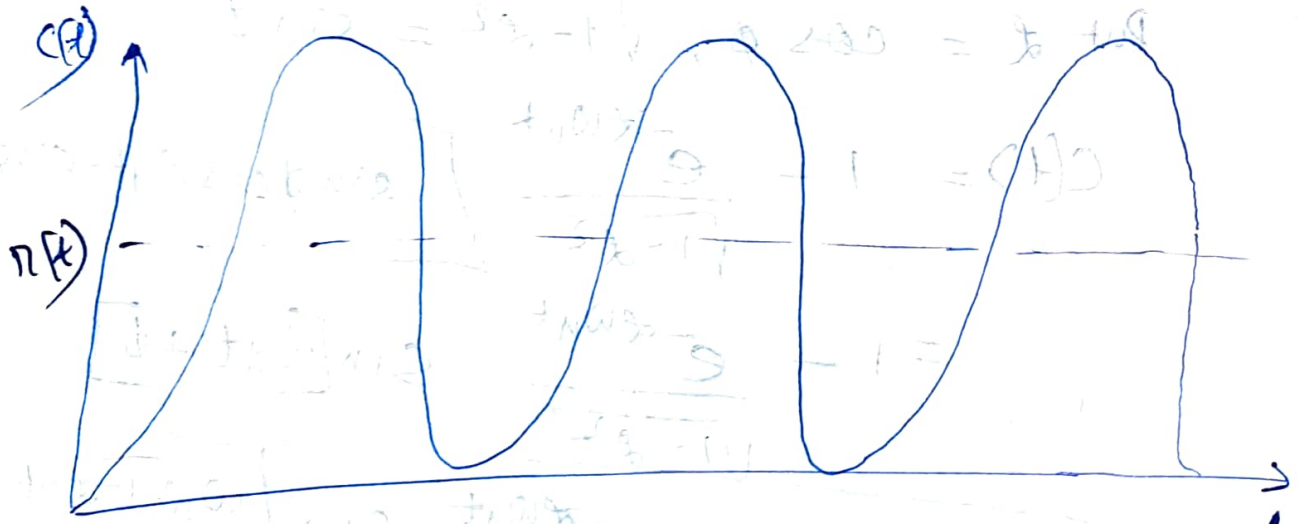
$\zeta\omega_n \rightarrow$ Damping factor.

$\omega_d \rightarrow$ Damped frequency of oscillation

Case 1

If $\zeta = 0$.

$$c(t) = 1 - \sin(\omega_n t + \tan^{-1}\infty)$$
$$= 1 - \sin(\omega_n t + \pi/2)$$
$$= 1 - \cos(\omega_n t)$$



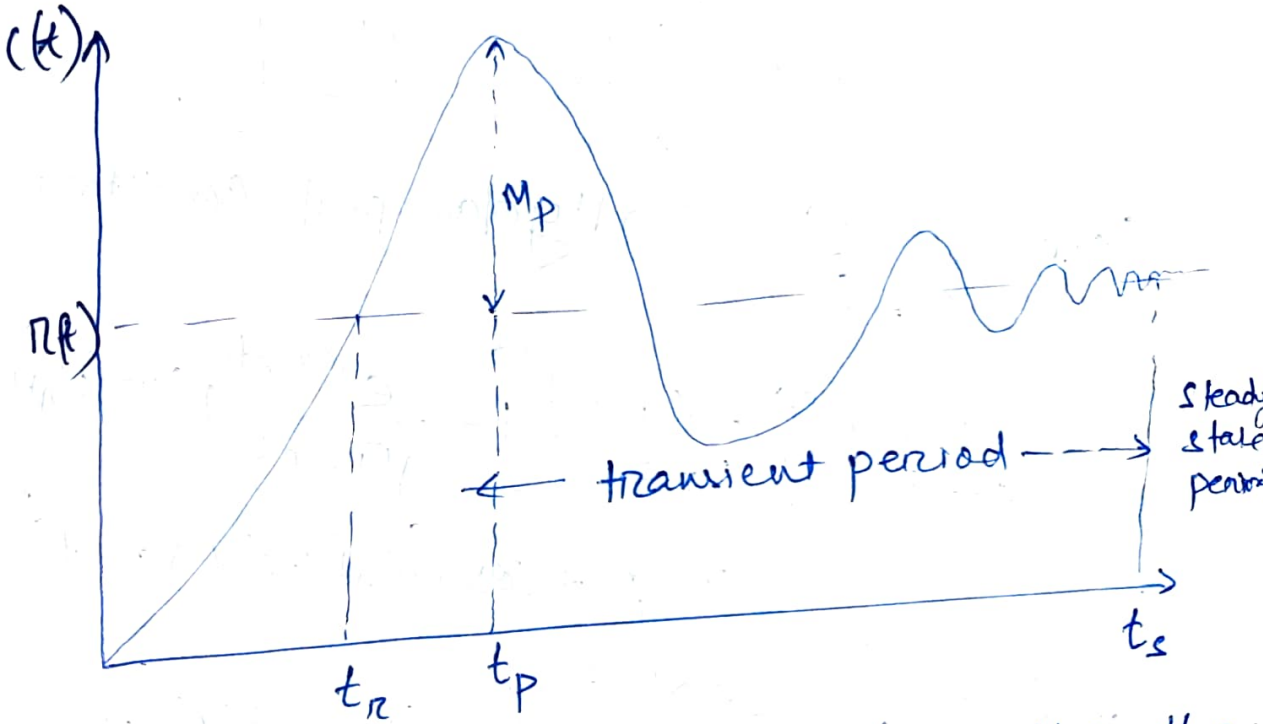
which is a sinosoidal with frequency ω_n.

So if ζ becomes zero the output exibit a sustain oscillation.

Case 2 Underdamped Condition.

$\zeta < 1$.

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \phi\right)$$



If ζ has a value of less than one, then the out put response of a second order control system becomes a exponentially decaying oscillation with a oscillating frequency of ω_d and a time constant of exponential decay of $1/\zeta\omega_n$.

$t_r \rightarrow$ Rise time: It is the time taken by the response to reach the desired value.

$M_p \rightarrow$ Maximum Overshoot: It is the maximum positive deviation from the desired value. It occurs at peak time t_p.

t_s : Settling time : It is the time required by the response to reach the steady state value. Normally the settling time is four times the time constant. i.e $\boxed{t_s = \frac{4}{\omega_n \zeta}}$.

Case 3 Critically damped condition

$\zeta = 1$.

$$\lim_{\zeta \to 1} c(t) = \lim_{\zeta \to 1} \left[1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \left\{ \sin\left(\omega_n \sqrt{1-\zeta^2}\, t\right) \cos\phi + \cos\left(\omega_n \sqrt{1-\zeta^2}\, t\right) \sin\phi \right\} \right].$$

$$= \lim_{\zeta \to 1} \left[1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \left\{ \sin\left(\omega_n \sqrt{1-\zeta^2}\, t\right) \zeta + \cos\left(\omega_n \sqrt{1-\zeta^2}\, t\right) \sqrt{1-\zeta^2} \right\} \right]$$

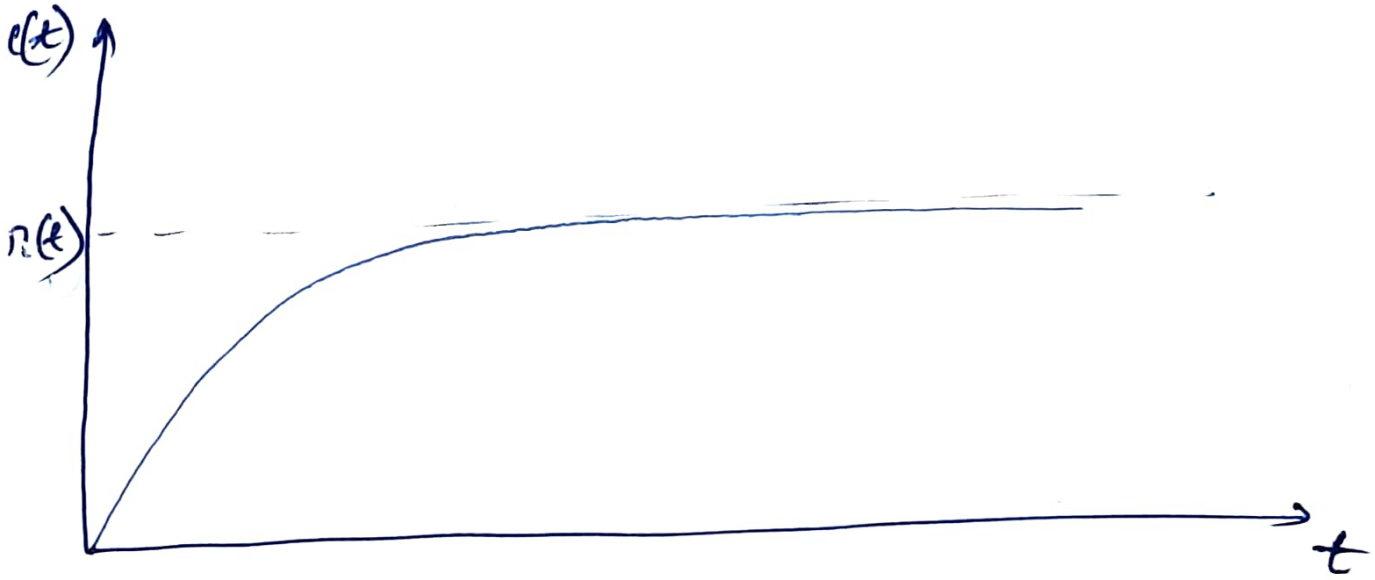
as $\lim_{\zeta \to 1} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t\right) = \omega_n \sqrt{1-\zeta^2}\, t$

and $\lim_{\zeta \to 1} \cos\left(\omega_n \sqrt{1-\zeta^2}\, t\right) = 1$.

$$\lim_{\zeta \to 1} c(t) = \lim_{\zeta \to 1} \left[1 - \frac{e^{-\zeta \omega_n t}}{1-\zeta^2} \left\{ \omega_n \sqrt{1-\zeta^2}\, t\, \zeta + \sqrt{1-\zeta^2} \right\} \right]$$

$$= \lim_{\zeta \to 1} \left[1 - e^{-\zeta \omega_n t} \left(\omega_n t\, \zeta + 1 \right) \right]$$

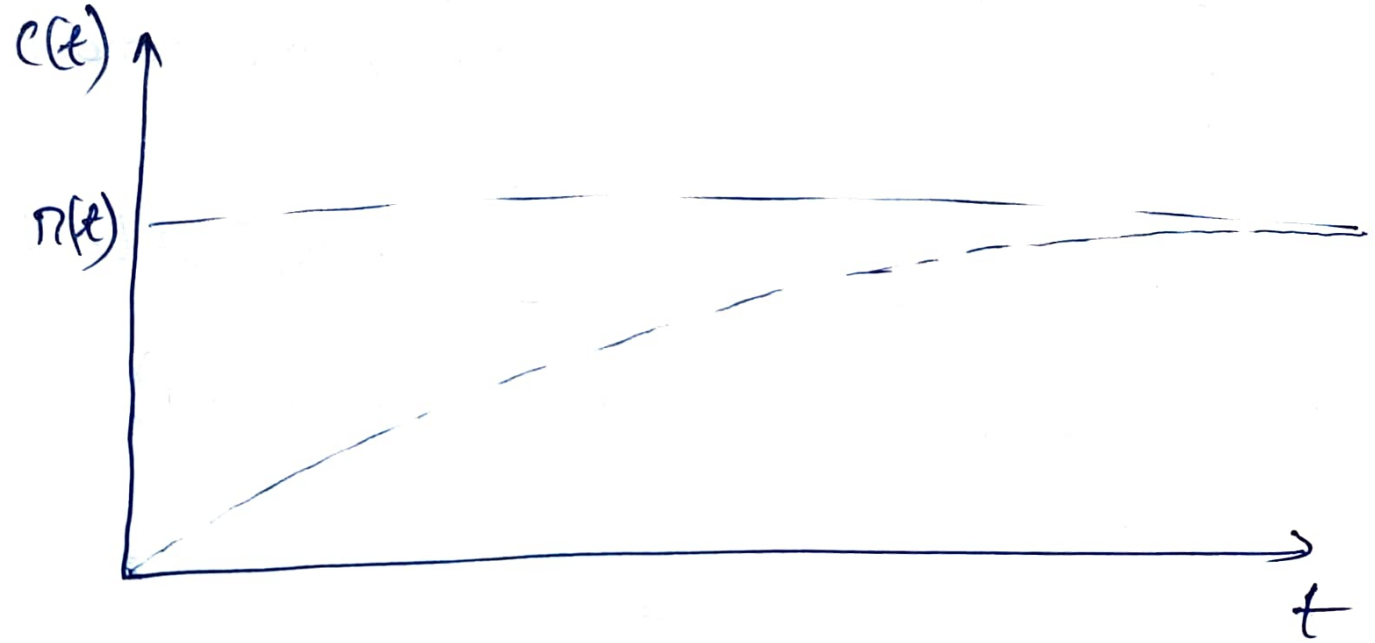
$$\Rightarrow \lim_{\zeta \to 1} c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



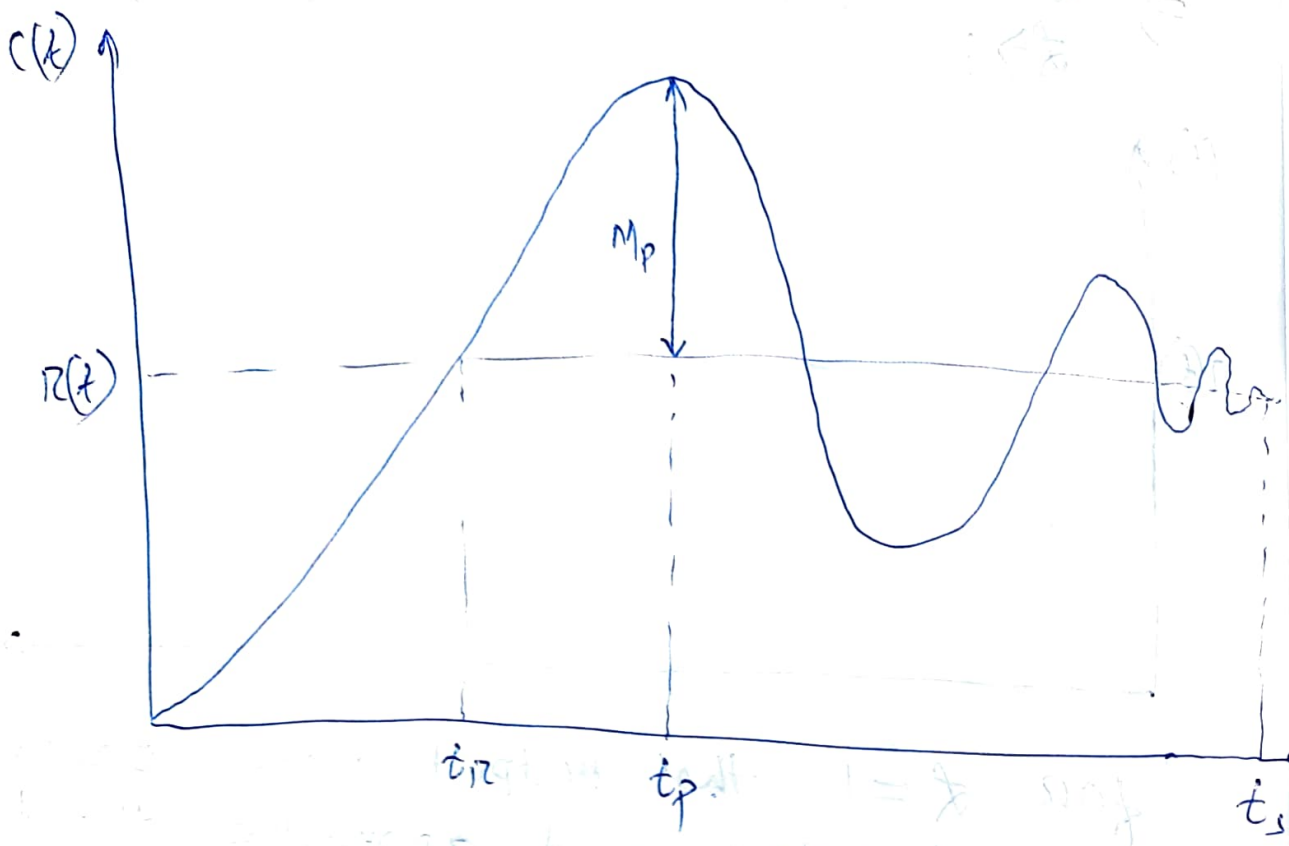
for $\zeta = 1$ the output never cross the desired value and approaches the desired value when $t = \infty$.

Case 4 Overdamped condition

$\zeta > 1$.



Transient Response Specification:-



Rise time (t_r):-

It is the time needed by the response to reach from 0 to 100% of the desired value for underdamped condition and 10 to 90% of desired value for overdamped condition.

Derivation

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\omega_n\sqrt{1-\zeta^2}\, t + \phi\right\}$$

For underdamped condition ($\zeta < 1$),

at $t = t_r$

$$c(t) = 1.$$

$$\therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \sin\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\} = 0.$$

$$\Rightarrow \sin\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\} = 0.$$

as $e^{-\zeta\omega_n t/\sqrt{1-\zeta^2}}$ is finite and nonzero.

The general solution is.

$$\omega_n\sqrt{1-\zeta^2}\,t_n + \phi = n\pi.$$

for $t = t_r$, $n = 1$.

$$\omega_n\sqrt{1-\zeta^2}\,t_r + \phi = \pi.$$

$$\therefore t_r = \frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi-\phi}{\omega_d}$$

$$= \frac{\pi - \tan^{-1}\left\{\frac{\sqrt{1-\zeta^2}}{\zeta}\right\}}{\omega_n\sqrt{1-\zeta^2}}$$

Maximum Overshoot (M_p)

The maximum positive deviation of the response from the desired value is called as maximum overshoot and it is denoted in percentage.

Peak time t_p

The time needed to reach the response to its peak value is called as peak time (t_p).

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left\{\omega_n\sqrt{1-\zeta^2}\,t + \phi\right\}$$

We can find the maxima by equating it first derivative to zero and find the solution.

$$\frac{d\,c(t)}{dt} = 0.$$

$$\Rightarrow -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\,\omega_n\sqrt{1-\zeta^2}\cos\left\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\right\}$$

$$+\zeta\omega_n\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\right\} = 0.$$

$$\Rightarrow \sqrt{1-\zeta^2}\cos\left\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\right\} = \zeta\sin\left\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\right\}$$

$$\Rightarrow \tan\left\{\omega_n\sqrt{1-\zeta^2}\,t+\phi\right\} = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

$$\Rightarrow \tan\left\{\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right\} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

The general solution is.

$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi$$

$$\Rightarrow t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \{n = 0, 1, 2 \cdots$$

When $n = 1$, $t = t_p$.

$$\therefore \boxed{t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}}$$

$$M_p = (-)\, e(t_p) = c(t_p) - 1$$

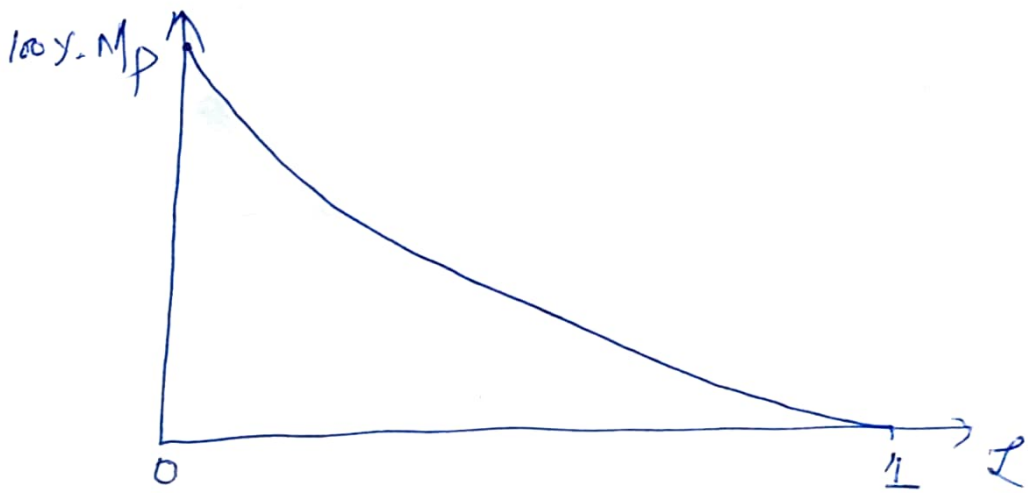
$$= (-)\frac{e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \sin\{\omega_d t_p + \phi\}$$

$$= (-)\frac{e^{-\frac{\pi \zeta \omega_n}{\omega_n \sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\left\{\omega_n \sqrt{1-\zeta^2}\,\frac{\pi}{\omega_n \sqrt{1-\zeta^2}} + \phi\right\}$$

$$= (-)\frac{e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin\{\pi + \phi\}$$

$$= \frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin\phi$$

$$\Rightarrow \boxed{M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}} \quad \{\text{as } \sin\phi = \sqrt{1-\zeta^2}\}$$



Settling time t_s

For 2%. $\rightarrow$ $t_s = \frac{4}{\zeta \omega_n}$

For 5%. $\rightarrow$ $t_s = \frac{3}{\zeta \omega_n}$

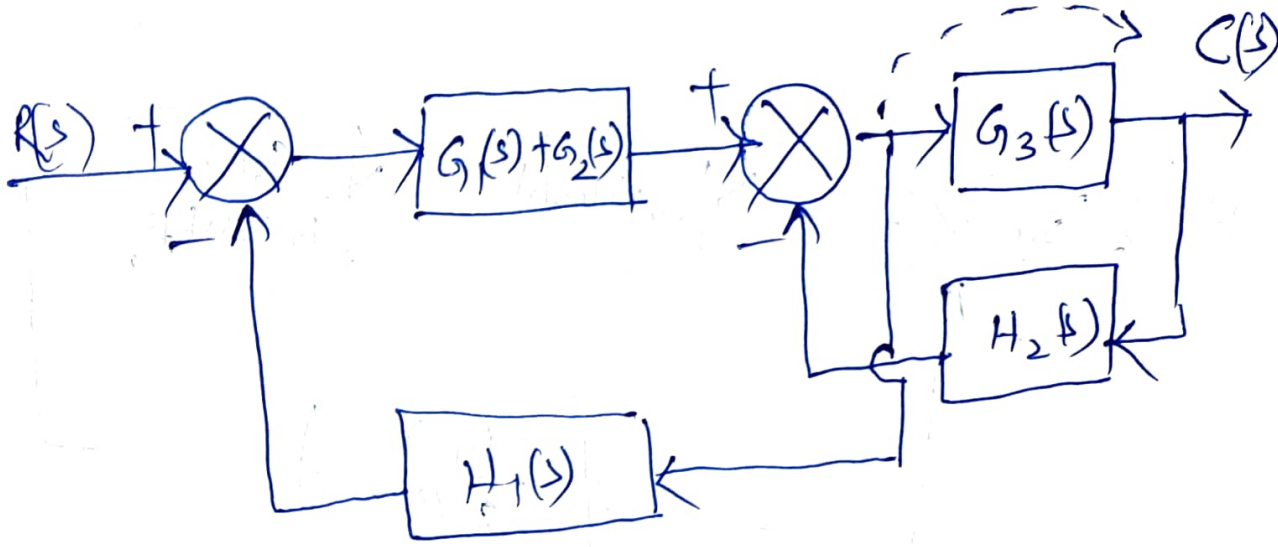
Procedure for Block diagram Reduction

Step 1. Reduce any cascade block, parallel block or loop present in the complex block diagram.

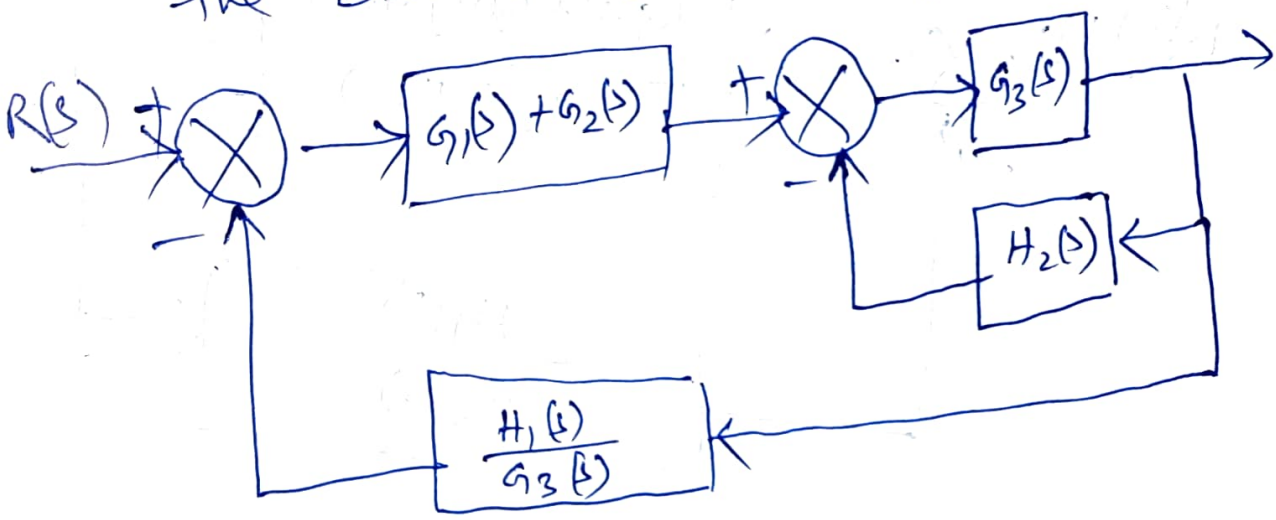
Step 2. Shift a take up point against a block so as to make a cascade block, parallel block or loop and repeat from step number one

Step 3. Shift a summing point against a block so as to make a cascade block, parallel block, or loop and repeat from step number one.

Step 4. ~~Shift a take up point against a summing point so as to make~~ Merge, demerge or interchange summing points so as to make ----.



As there are no more cascade, parelle or simple loop, we have to move the take up point before the block $G_3(s)$ to after the block.

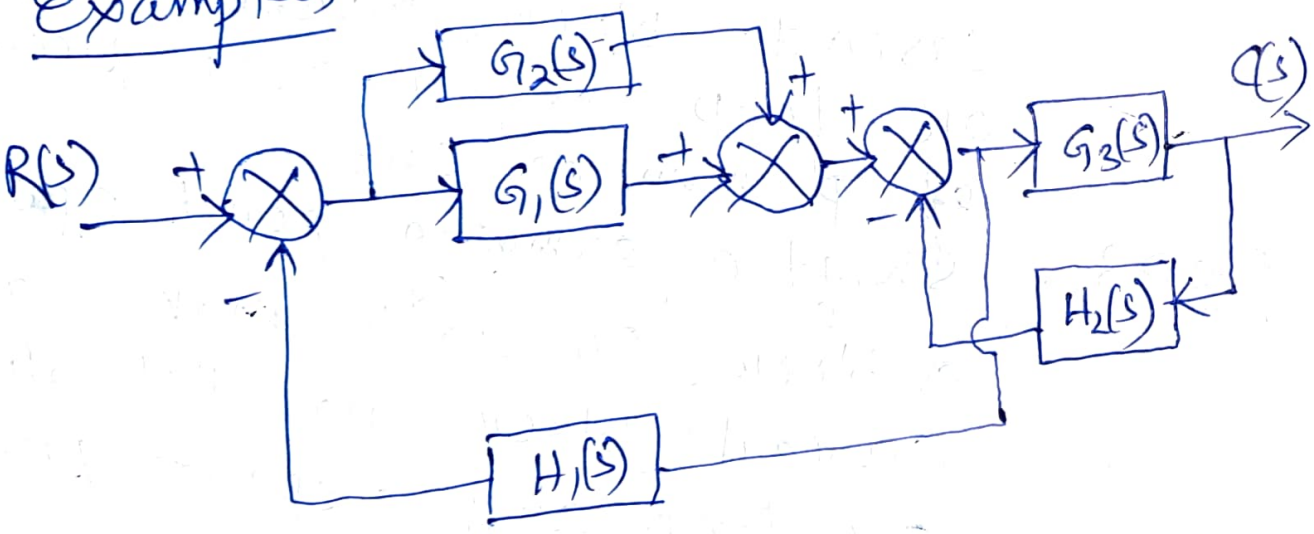


We found a simple ~~block~~ loop comprise of $G_3(s)$ and $H_2(s)$. So we reduce that.

Step 5. If necessary then shift a take up point against a summing point so as to ——————.

—X—

Examples



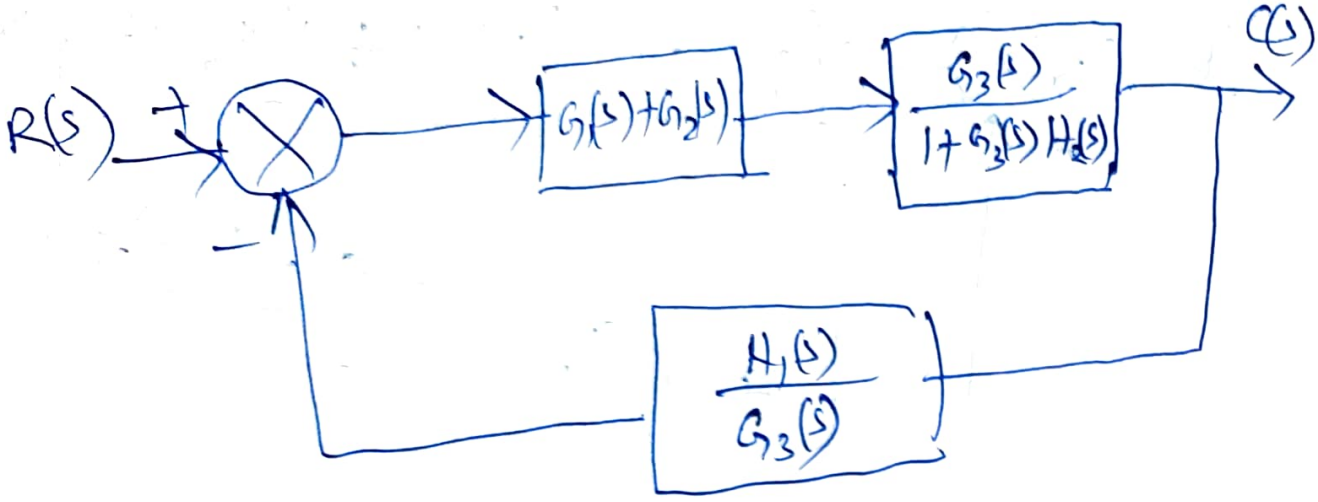
As there is a parallel blocks $G_1(s)$ and $G_2(s)$, we reduce the same first.

$$\frac{C(s)}{R(s)} = \frac{\dfrac{[G_1(s)+G_2(s)]\,G_3(s)}{1+G_3(s)H_2(s)}}{1+\dfrac{[G_1(s)+G_2(s)]\,G_3(s)}{1+G_3(s)H_2(s)} \times \dfrac{H_1(s)}{G_3(s)}}$$

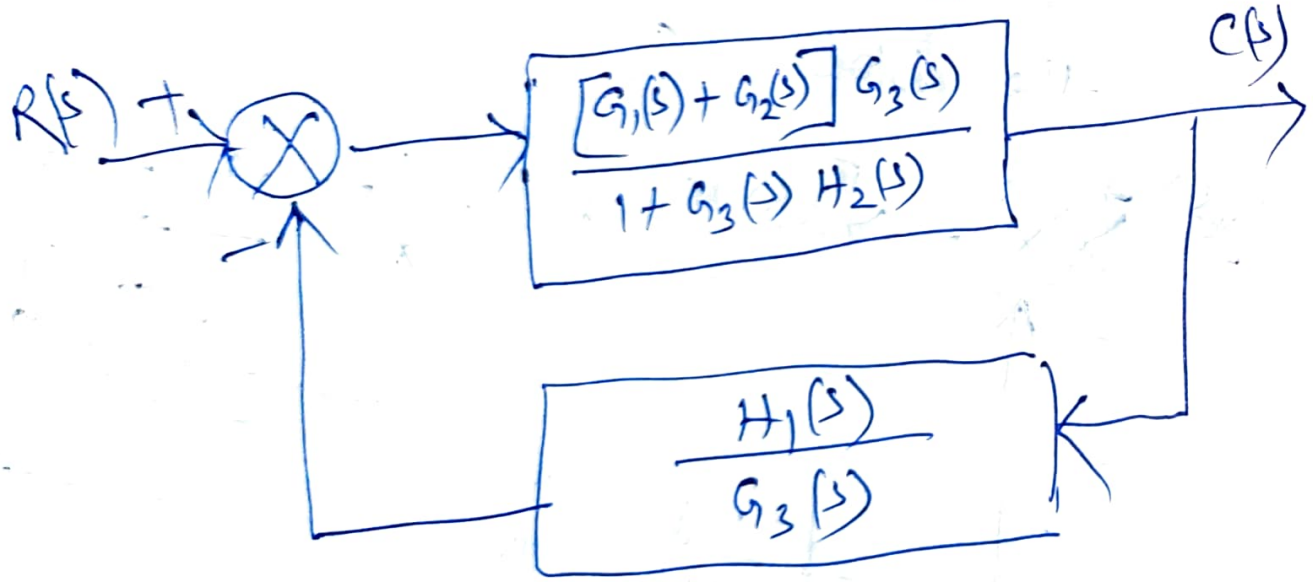
$$= \frac{G_1(s)G_3(s)+G_2(s)G_3(s) \,/\, 1+G_3(s)H_2(s)}{1+G_3(s)H_2(s)+G_1(s)H_1(s)+G_2(s)H_1(s) \,/\, 1+G_3(s)H_2(s)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1(s)G_3(s)+G_2(s)G_3(s)}{1+G_1(s)H_1(s)+G_2(s)H_1(s)+G_3(s)H_2(s)}}$$

Ans



Now there is a cascade block.



No we redwe the loop.

Mason's Gain Formula

$$T = \sum_{K=1}^{K} \frac{\Delta_K P_K}{\Delta}$$

K = number of forward path

P_K = K^{th} forward path Gain.

Δ = 1 − (Summation of all loop gain)

+ (summation of product of all non touching loop pairs)

− (summation of product of all non touching loop triplets)

+ - - - - .

Δ_K = It is apart of Δ which are non touching to the forward path K.

Signal Flow Graph.

Terms

Node :- It is a point representing a signal at a particular position.

Branch :- It is a connection of two nodes having a direction and its associated gain.

Path :- It is a combination of Branch where the signal can flow.

Forward Path :- It is a path starting from the input node and terminates at output node, without repeating any intermediate nodes.

loop :- It is a path starts and terminates at same node without repeating any other node.

non touching loop :- These are two or more loops which are non-touching to each other.