FLUID KINEMATICS

* Fluid Kinematics -

The branch of science which deals with the motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in fluid kinematics.

→ Once the velocity is known, then the pressure distribution and hence the forces acting on the fluid can be determined.

* Methods of describing fluid Motion :-

(i) Lagrangian Method - A single fluid particle is followed during its motion and its velocity, acel^n, density etc is described.

(ii) Eulerian Method - The velocity, acel^n, pressure, density etc. are described at a point in the flow field, in this method. It deals with the study of flow pattern of all particles ~~at~~ simultaneously at one section. In this method the path traced by all the particles at one section and at one time are studied in detail.

→ Eulerian method is commonly used in fluid mechanics due to its mathematical simplicity.

* Flow Lines -

Whenever a fluid is in motion, the fluid particles move along certain lines, depending upon the characteristic of the fluid and the shape of the passage through which the fluid particles move. These lines are called "Flow lines"

(i) Path Lines (ii) Stream lines

(iii) Streak lines or Filament lines

(iv) Potential or Equipotential lines.

(i) Path Lines - The path followed by a fluid particle in motion is called path line. Thus the path line shows the direction of a particle for a certain period of time or between two given sections.

(ii) Stream lines - The imaginary line, drawn in the fluid in such a way that the tangent to any point gives the direction of motion at the point. The stream line shows the direction of motion of a number of particles at the same time. There is no movement of fluid across a streamline.

→ Stream tube - An fluid element, bonded by a number of stream lines, which confine the flow is called stream tube. No fluid ~~entre~~ can entre or leave the stream tube except at the ends.

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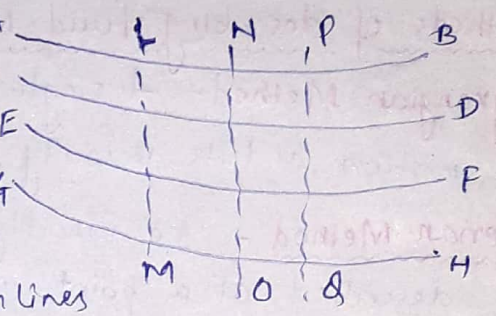
(iii) Streak Lines or Filament Lines :- The instantaneous pictures of the position of all fluid particles, which ~~behaves~~ have passed through a given point at some previous time are called streak lines or filament lines. e.g the lines formed by smoke particles ejected from a nozzle is a streak line.

(iv) Potential Lines or Equipotential Lines :-

We know that there is always a loss of head of the fluid particles, as we proceed along the flow lines. If we draw lines joining the points of equal potential on adjacent flow lines, we get the equipotential lines.



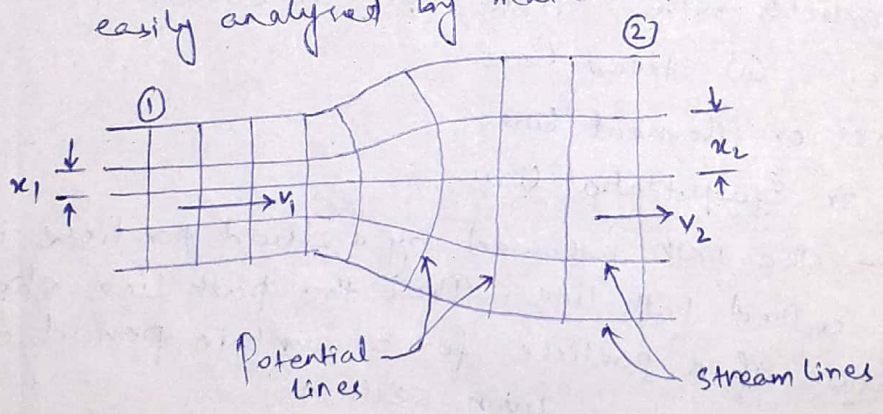
AB, CD, EF, GH are stream lines

LM, NO, PQ are potential lines.

* Flow Net and Its Uses :-

If we draw both stream lines and potential lines for a flow, the pattern, obtained by the inter-section of the two sets of lines is called the flow net.

→ flow net helps us in analysing the behaviour of certain fluid phenomenon which cannot be easily analysed by mathematical means.



Since there is no flow across the streamlines, therefore discharge per unit width, between two ~~consequ~~ consecutive stream lines will be equal.

Therefore discharge per unit width (q) :-

$$\boxed{q = x_1 V_1 = x_2 V_2}$$

Thus, the velocity of fluid particles decreases with the increase in spacing and vice versa.

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(Q) From a flownet diagram, it is found that the distance between two consecutive streamlines at two successive sections are 10 mm and 6 mm respectively. If the velocity at the first section is 1 m/s, find the velocity at the other section. Also find the discharge between these two stream-lines.

Sol^n: $x_1 = 10\,mm = 0.01\,m$ $\quad V_1 = 1\,m/s$ $\quad q = ?$

$x_2 = 6\,mm = 0.006\,m$ $\quad V_2 = ?$

$$q = x_1 V_1 = x_2 V_2 \Rightarrow V_2 = \frac{x_1 V_1}{x_2} = \frac{0.01 \times 1}{0.006} = 1.67\ m/s$$

$$\therefore q = x_1 V_1 = 0.01 \times 1 = 0.01\ m^3/sec = 10\ lit/sec \quad (\because 1\,m^3 = 1000\,lit)$$

$= 10$ lit./sec (Ans)

* Types of fluid flow :-

(i) Steady and Unsteady flow
(ii) Uniform and Non-Uniform flow
(iii) Laminar and Turbulent flow
(iv) Compressible and Incompressible flow
(v) Rotational and Irrotational flow
(vi) One, Two and Three-dimensional flow

(i) Steady and Unsteady flows :-

→ Steady flow is that type of flow in which the fluid characteristics like velocity, pressure, density etc. at a point do not change with time. i.e

$$\left(\frac{dv}{dt}\right)_{x_0,y_0,z_0} = 0, \quad \left(\frac{dp}{dt}\right)_{x_0,y_0,z_0} = 0, \quad \left(\frac{d\rho}{dt}\right)_{x_0,y_0,z_0} = 0$$

where, (x_0, y_0, z_0) is a fixed point in fluid field.

→ Unsteady flow is that type of flow in which the velocity, pressure, density at a point changes w.r.t to time.

$$\left(\frac{dv}{dt}\right)_{x_0,y_0,z_0} \neq 0, \quad \left(\frac{dp}{dt}\right)_{x_0,y_0,z_0} \neq 0, \quad \left(\frac{d\rho}{dt}\right)_{x_0,y_0,z_0} \neq 0$$

(ii) Uniform and Non-uniform flows :-

→ Uniform flow is that type of flow in which the velocity at any given time does not change w.r.t space (i.e length of direction of flow). i.e $\boxed{\left(\frac{dv}{ds}\right)_{t=const.} = 0}$

dv = change of velocity

ds = length of flow in the direction of s.

→ Non-uniform flow = velocity at any given time changes w.r.t space. i.e $\boxed{\left(\frac{dv}{ds}\right)_{t=const.} \neq 0}$

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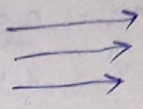
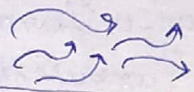
(iii) Laminar and Turbulent flow :-

→ Laminar flow is that type of flow in which the fluid particles moves along well defined paths or stream line and all stream lines are straight and parallel. This flow is also called "stream-line flow" or "Viscous flow".



$Re < 2000$

→ Turbulent flow is that type of flow in which the fluid particles move in zig-zag way. Due to the movement of fluid particles in a zig-zag way, eddies formation takes place which is responsible for high energy loss.



$Re > 4000$

→ Reynold's Number $\boxed{Re = \frac{VD}{\nu}}$ This is a dimensionless no. used to determine the type of flow in a pipe.

If Re lies between 2000 and 4000 then the flow may be laminar or turbulent.

D = dia. of pipe
V = velocity of flow in pipe
ν = Kinematic viscosity of fluid

(iv) Compressible and Incompressible flows :-

→ Compressible flow - the flow in which the density of the fluid changes from point to point. i.e $\rho \neq$ constant. (Gas flow are compressible flow)

→ Incompressible flow - the flow in which the density of fluid doesnot changes from point to point. i.e $\rho =$ constant. (Liquids are generally incompressible)

(v) Rotational and Irrotational flow :-

→ Rotational flow is the flow in which the fluid particles while flowing along stream-lines also rotate about their own axis.

→ Irrotational flow - the flow in which the fluid particles while flowing through stream-lines do not rotate about their own axis.

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vi) One-, Two- and Three-Dimensional Flows :-

→ One dimensional flow is the flow in which the flow parameter such as velocity is a function of time and one space-co-ordinate only say x.

i.e $u = f(x)$, $v = 0$ and $w = 0$

where, u, v, and w are velocity components in x, y and z directions respectively.

→ Two dimensional flow - the flow in which velocity is a function time and two rectangular space co-ordinates say x and y.

$$u = f_1(x, y)$$
$$v = f_2(x, y)$$
$$w = 0$$

→ Three dimensional flow - the flow in which velocity is a function of time and three mutually perpendicular directions say, x, y and z.

$$u = f_1(x, y, z)$$
$$v = f_2(x, y, z)$$
$$w = f_3(x, y, z)$$

Rate of Discharge or Rate of flow (Q) :-

→ The quantity of a fluid flowing per second through a section of pipe or a channel. for incompressible fluid 'Q' is expressed in volume flow rate. m^3/s or l/s

→ In compressible fluid 'Q' is expressed in weight of fluid flowing across section. kgf/sec or N/sec.

$$Q = A \times V$$

[A = Cross sectional area of pipe
V = Avg. velocity of fluid across the section.

Continuity Equation :-

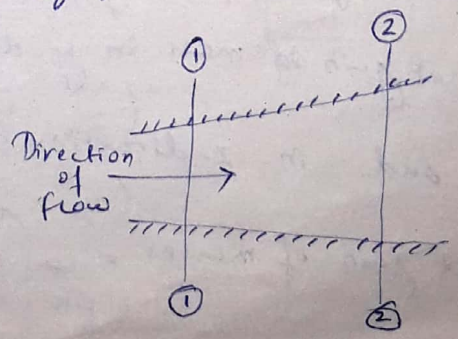
The equation based on the principle of conservation of mass is called Continuity Eqn.

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

→ for incompressible flow

$$\rho_1 = \rho_2$$

$$\therefore \boxed{A_1 V_1 = A_2 V_2}$$



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Example (8) A 30 cm dia. pipe conveying water branches into two pipes of dia 20 cm and 15 cm respectively. If avg. velocity in 30 cm dia. pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the avg. velocity in 20 cm dia pipe is 2 m/s.

Sol^n:

$D_1 = 30\,cm = 0.3\,m$

$A_1 = \frac{\pi}{4} D_1^2 = 0.07068\,m^2$

$V_1 = 2.5\,m/s$

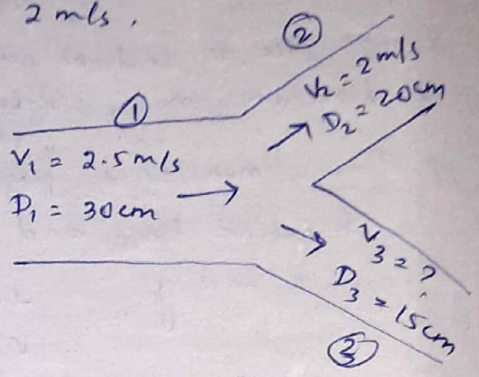
$D_2 = 20\,cm = 0.2\,m$

$A_2 = \frac{\pi}{4} D_2^2 = 0.0314\,m^2$

$V_2 = 2\,m/s$

$D_3 = 15\,cm = 0.15\,m$

$A_3 = \frac{\pi}{4}(0.15)^2 = 0.01767\,m^2$



According to continuity eq^n $Q_1 = Q_2 + Q_3$

$$\Rightarrow A_1V_1 = A_2V_2 + A_3V_3$$

$$\Rightarrow A_3V_3 = A_1V_1 - A_2V_2$$

$$\Rightarrow V_3 = \frac{A_1V_1 - A_2V_2}{A_3}$$

$$\Rightarrow V_3 = \frac{0.07068 \times 2.5 - 0.0314 \times 2}{0.01767}$$

$$\Rightarrow V_3 = 6.44\,m/s \quad \text{(Ans)}$$

* Continuity Eq^n in three-dimensions :-

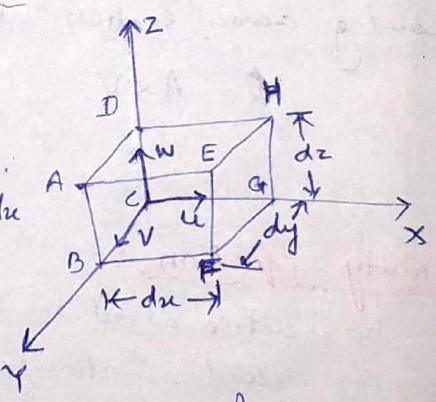
Gain of mass in x-direction

= (Mass through ABCD) − (Mass through EFGH) per sec.

$= \rho u\,dy\,dz - \rho u\,dy\,dz - \frac{\partial}{\partial x}(\rho u\,dy\,dz)\,dx$

$= -\frac{\partial}{\partial x}(\rho u\,dy\,dz)\,dx$

$= -\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz$



Similarly net gain of mass in y-direction $= -\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz$

and in z-direction $= -\frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz$

$\therefore$ Net gain of masses $= -\left[\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right)\right] \times dx\,dy\,dz$

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According principle of conservation of mass :-

"Net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element."

If mass of fluid in element is $\rho\, dx.dy.dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\, dx dy dz)$ or

$$\frac{\partial \rho}{\partial t} dx dy dz \quad \text{--- (2)}$$

$\therefore$ Equating eq^n (1) and (2)

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \times dx dy dz = \frac{\partial \rho}{\partial t} dx dy dz$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0} \quad \text{--- (3)}$$

(i) for steady flow $\frac{\partial \rho}{\partial t} = 0$

$\therefore$ eq^n (3) $\Rightarrow \boxed{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$

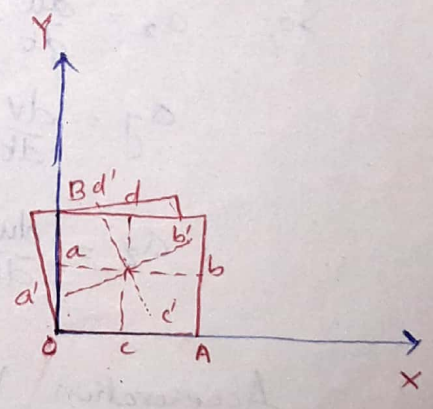
(ii) for incompressible flow ρ = constant

$\therefore$ eq^n (3) $\Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$ --- (4)

(iii) for two dimensional incompressible flow,

$w = 0$ $\therefore$ eq^n (4) $\Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$

Mathematical Definition of Rotational and Irrotational flow :-



Pure Rotation - The movement of a fluid element in such a way that both of its axes rotate in same direction.

For two Dimensional Rotational flow,

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

If $\omega_x = \omega_y = \omega_z = 0$, Irrotational flow

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* Velocity and Acceleration :-

→ If velocity components are given as :-

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity $V = ui + vj + wk$

$$= \sqrt{u^2 + v^2 + w^2}$$

By Chain rule of differentiation :-

→ Let a_x, a_y and a_z are total acceleration in x, y, and z directions respectively then :-

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt} + \frac{\partial u}{\partial z} \times \frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But, $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$

$$\therefore \quad a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = \frac{du}{dt}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = \frac{dv}{dt}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = \frac{dw}{dt}$$

→ for Steady Flow, $\frac{dV}{dt} = 0$ $\quad V$ = Resultant velocity

or $\frac{\partial u}{\partial t} = 0$, $\frac{\partial v}{\partial t} = 0$, $\frac{\partial w}{\partial t} = 0$

So,

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Accel^n for Steady flow

Acceleration Vector $A = a_x i + a_y j + a_z k$

$$= \sqrt{a_x^2 + a_y^2 + a_z^2}$$

(* Note :- For steady flow, the local accelerations is zero or negligible.)

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→ **Local Acceleration** - It is defined as the rate of increase of velocity w.r.t time at a given point in a flow field. i.e $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as Local Acc^n.

→ **Convective Acceleration** :- It is defined as the rate of change of velocity due to change of position of particles in a fluid flow is known as Convective acceleration.

e.g. $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = a_x$ = Convective acc^n component in x-direction.

2013
5 marks

(Q) In a 3-dimensional incompressible fluid flow field is given by expression :-
$V = (x^2+y^2z^3)i - (xy+yz+zx)j + (w)k$. find the w component of a velocity so that the case is possible for a steady incompressible fluid flow.

Ans:- for steady in-compressible flow, the continuity eq^n is :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{——— (1)}$$

Here, $u = x^2+y^2z^3$
$v = -(xy+yz+zx) = -xy-yz-zx$
$w = ?$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2z^3) = 2x$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-xy-yz-zx) = -x-z$

So, eq^n (1) can be written as :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow 2x - x - z + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = z - x$$

$$\Rightarrow \int \partial w = \int (z-x)\,\partial z$$

$$\Rightarrow w = \frac{z^2}{2} - xz + C$$, C = constant of integration

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The constant of integration C cannot be a function of z. So, $c = f(x,y)$

$$\therefore \boxed{w = \frac{z^2}{2} - xz + f(x,y)} \quad \underline{\underline{\text{Ans}}}$$

Assignment (Q) A fluid flow field is given by :-
(10 marks)

$$V = x^2y\,i + y^2z\,j - (2xyz + yz^2)\,k$$

Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2,1,3).

Sol^n: $u = x^2y$ $\qquad \frac{\partial u}{\partial x} = 2xy$

$v = y^2z$ $\qquad \frac{\partial v}{\partial y} = 2yz$

$w = -2xyz - yz^2$ $\qquad \frac{\partial w}{\partial z} = -2xy - 2yz$

for steady incompressible flow :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow 2xy + 2yz - 2xy - 2yz = 0 \quad \text{(Proved)}$$

→ Velocity at (2,1,3) $\quad x=2,\ y=1,\ z=3$

$$\therefore V = x^2y\,i + y^2z\,j - (2xyz + yz^2)\,k$$

$$= (2^2\times1)\,i + (1^2\times3)\,j - (2\times2\times1\times3 + 1\times3^2)\,k$$

$$= 4i + 3j - 21k$$

So, Resultant velocity $V = \sqrt{4^2 + 3^2 + (-21)^2}$

$$= \sqrt{16 + 9 + 441}$$

$$= \sqrt{466} = 21.587 \text{ units.}$$

→ Acceleration at (2,1,3), $\frac{\partial u}{\partial t} = 0,\ \frac{\partial v}{\partial t} = 0,\ \frac{\partial w}{\partial t} = 0$ (for steady flow)

$$\therefore a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= x^2y \times 2xy + y^2z \times x^2 + (-2xyz - yz^2)\times 0$$

$$= 2x^3y^2 + x^2y^2z$$

$$= 2\times2^3\times1^2 + 2^2\times1^2\times3$$

$$= 16 + 12 = 28 \text{ units}$$

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$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= x^2y \times 0 + y^2z \times 2yz + (-2xyz - yz^2) \times y^2$$

$$= 2y^3z^2 - 2xy^3z - y^3z^2$$

$$= y^3z^2 - 2xy^3z$$

$$= 1^3 \times 3^2 - 2 \times 2 \times 1^3 \times 3 = 9 - 12 = -3 \text{ units}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= x^2y \times (-2yz) + y^2z \times (-2zx) + (-2xyz - yz^2) \times (-2xy - 2yz)$$

$$= -24 - 36 - 27 + 48 + 36 + 72 + 54$$

$$= 123 \text{ units}$$

$\therefore$ Acceleration $\Rightarrow A = a_x i + a_y j + a_z k$

$$= 28i - 3j + 123k$$

Resultant acc^n $A = \sqrt{28^2 + (-3)^2 + (123)^2}$

$$= \sqrt{15922}$$

$$= 126.18 \text{ units.}$$

* Vorticity :- It is defined as the value twice of the rotation.

$\therefore$ Vorticity $= 2\omega$ or $2\omega_x$, $2\omega_y$, $2\omega_z$ in x, y, and z direction

* **Velocity Potential function (ϕ)** It is a scalar function of space and time such that its negative derivative w.r.t any direction gives the fluid velocity in that direction.

$$u = -\frac{\partial \phi}{\partial x} \qquad v = -\frac{\partial \phi}{\partial y} \qquad w = -\frac{\partial \phi}{\partial z}$$

→ For an incompressible steady flow, the continuity eqⁿ :-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \rightarrow \text{Laplace Eqⁿ}$$

→ For 2-D flow $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow$ Laplace eqⁿ

→ Properties of velocity potential function (ϕ) are :-

(i) If velocity potential (ϕ) exists, the flow should be irrotational i.e $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right)$

$$\Rightarrow \omega_z = 0$$

(ii) If velocity potential (ϕ) satisfies Laplace Eqⁿ, it represents the possible steady incompressible irrotational flow.

* **Stream function (ψ)** :- It is the scalar function of space and time such that partial derivative w.r.t any direction gives the velocity component at right angles to that dirⁿ.

→ For steady flow $\psi = f(x, y)$, $\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\}$

→ Continuity eqⁿ for 2-D flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\Rightarrow -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

→ Properties of stream function (ψ) :-

(i) If stream function (ψ) exists, it is a possible case of an irrotational flow or rotational flow.

(ii) If ψ satisfies the Laplace eqⁿ, it is a possible case of an irrotational flow.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (\text{for } \omega_z = 0)$$

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* **Equipotential Line** :- A line along which the velocity potential ϕ is constant is called "equipotential line".

$\therefore \phi = \text{constant}$

$\phi = f(x, y)$ for steady flow

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \qquad \left\{ \because \frac{\partial \phi}{\partial x} = -u, \ \frac{\partial \phi}{\partial y} = -v \right\}$$

$$\Rightarrow d\phi = -u\,dx - v\,dy$$

for Equipotential lines $d\phi = 0$

$$\Rightarrow -u\,dx - v\,dy = 0$$

$$\Rightarrow -(u\,dx + v\,dy) = 0$$

$$\Rightarrow u\,dx + v\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{u}{v} = \text{Slope of Equipotential line}$$

* **Line of Constant Stream function** :-

$\psi = \text{constant} \Rightarrow d\psi = 0$

$$\Rightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \qquad \left\{ \because \frac{\partial \psi}{\partial x} = v, \ \frac{\partial \psi}{\partial y} = -u \right\}$$

$$\Rightarrow v\,dx - u\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{v}{u} = \text{Slope of stream line}$$

2015 5 marks

(Q) In 2-D incompressible flow, the fluid velocity components are $u = x - 4y$ and $v = -y - 4x$. Show that velocity potential exists and find its form as well as stream function.

Sol^n: $u = x - 4y$ $\qquad \frac{\partial u}{\partial x} = 1$

$v = -y - 4x$ $\qquad \frac{\partial v}{\partial y} = -1$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$ Hence flow is continuous and velocity potential exists.

Let ϕ = velocity potential

$$\therefore \frac{\partial \phi}{\partial x} = -u = -(x - 4y) = -x + 4y \quad \text{-----} \ (1)$$

$$\frac{\partial \phi}{\partial y} = -v = -(-y - 4x) = y + 4x \quad \text{-----} \ (2)$$

Integrating eq^n (1) $\quad \phi = -\frac{x^2}{2} + 4xy + c \quad \text{-----} \ (3)$

C is const. of integration and it is independent of x. But may be funⁿ of y.

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Differentiating eq^n ③ w.r.t y we get

$$\frac{\partial \phi}{\partial y} = 0 + 4x + \frac{\partial c}{\partial y} = y + 4x \quad \text{, from eq}^n \text{ ②}$$

$$\Rightarrow 4x + \frac{\partial c}{\partial y} = y + 4x \Rightarrow \frac{\partial c}{\partial y} = y \Rightarrow \int \partial c = \int y\,dy$$

$$\Rightarrow c = y^2/2 + c_1$$

$$\Rightarrow c = y^2/2 \text{ , if } c_1 = 0$$

$$\therefore \boxed{\phi = -\frac{x^2}{2} + 4xy + \frac{y^2}{2}} \quad \text{Ans}$$

→ Value of Stream function :-

$$\frac{\partial \psi}{\partial x} = v = -y - 4x \quad \text{— ④}$$

$$\frac{\partial \psi}{\partial y} = -u = -(x - 4y) = -x + 4y \quad \text{— ⑤}$$

Integrating eq^n ④ w.r.t x

$$\psi = -yx - \frac{4x^2}{2} + K \quad \text{— ⑥}$$

Differentiating eq^n ⑥ w.r.t y, we get

$$\frac{\partial \psi}{\partial y} = -x - 0 + \frac{\partial K}{\partial y} = -x + 4y \quad \text{, from eq}^n \text{ ⑤}$$

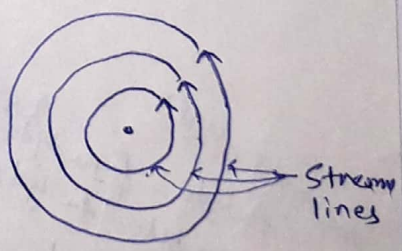
$$\Rightarrow -x + \frac{\partial K}{\partial y} = -x + 4y$$

$$\Rightarrow \frac{\partial K}{\partial y} = 4y \Rightarrow \int \partial K = \int 4y\,\partial y = 4\frac{y^2}{2} = 2y^2$$

$$\therefore \boxed{\psi = -yx - 2x^2 + 2y^2} \quad \text{Ans}$$

* Circulation (or free Vortex Flow)

free vortex flow is a Circulatory flow of a fluid such that its stream lines are concentric circles.



for free vortex $u_\theta \times r = C$

Circulation $\Gamma = 2\pi r \times u_\theta$

$= 2\pi C$

u_θ = tangential velocity at any radius r from centre

$$\therefore u_\theta = \frac{\Gamma}{2\pi r}$$

$u_r = 0$ radial velocity

Γ (Greek upper case gamma)

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* Vorticity is the circulation per unit area, taken around an infinitesimal loop. Correspondingly, the flux of vorticity is the circulation.

$$\Gamma = \oint_{ds} V \cdot dl = \iint_S \omega \cdot ds$$

if $\omega = \nabla \times S$

S = closed surface but not just a closed curve.

dl = line element along boundary

* In fluid dynamics, Circulation is the line integral around a closed curve of the velocity field.

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Remember:

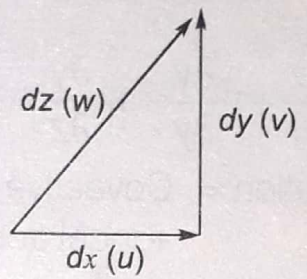
- ☑ There is no **velocity** component normal to stream lines.
- ☑ In steady flow Streakline, Pathline and Streamline are Identical.
- ☑ There is no discharge across a streamline.

7. Equation of stream line

Tangent to stream line gives velocity

$\vec{V} \times \vec{dx} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ dx & dy & dz \end{vmatrix} = 0$$



$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Here, u, v, w = Components of velocity in x, y, z direction

8. Continuity Equation (Conservation of Mass)

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad \text{for compressible flow}$$

Here, ρ = Density

A = Area

V = Velocity

- For incompressible fluid density will be constant thus continuity equation will be

$$A_1 V_1 = A_2 V_2 \quad \text{for incompressible flow}$$

9. General Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\frac{d\rho}{dt} + \rho(\nabla \vec{V}) = 0 \text{ (Vector form)}$$

Special Case:

- If flow is steady then $\left(\frac{\partial \rho}{\partial t} = 0\right)$

  Thus continuity equation will be

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

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- For steady, incompressible density will be constant thus

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Div. $\vec{V}=0$

10. Total Acceleration of fluid

$$a_x = \underbrace{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}}_{\text{Convective acceleration}}+\underbrace{\frac{\partial u}{\partial t}}_{\text{Temporal or local acceleration}}$$

$$a_y = u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}$$

- Total Acceleration = Covective acceleration with respect to space + local acceleration with respect to time

Type of flow	Convective Acceleration	Temporal Acceleration
Steady and uniform	0	0
Steady and non- uniform	Exists	0
Unsteady and uniform	0	Exists
Unsteady and non- uniform	Exists	Exists

Tangential acceleration - Magnitude of velocity changes
Normal acceleration - Direction of velocity changes

11. Rotational Component/Vorticity/Circulation

- Rotational component (ω)

$$\omega = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

For irrotational flow

$$\omega_x = \omega_y = \omega_z = 0$$

- Vorticity (ξ) : 2D, Scalar quantity

$\xi = 2 \times$ Rotational component

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- **Circulation (Γ)**
  It is line integral of tangential component of velocity around a closed curve.

$$\Gamma = \text{Vorticity} \times \text{Area}$$

$$\Rightarrow \text{Vorticity} = \frac{\Gamma}{A} \text{ [circulation density]}$$

Remember:

- ☑ In irrotational flow, the vorticity is zero, at all points in the flow region while for rotational flow, vorticity is non-zero.
- ☑ Flow outside the boundary layer has irrotational characteristic while that within the boundary layer is rotational characteristic.

12. Velocity Potential Function (ϕ)

$$u = \frac{-\partial\phi}{\partial x}, \quad v = \frac{-\partial\phi}{\partial y}, \quad w = \frac{-\partial\phi}{\partial z}$$

Remember:

- ☑ Flow always occurs in the direction of decreasing potential.
- ☑ Velocity potential function exists only for irrotational flow and potential flow.
- ☑ Laplace equation is given by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \boxed{\nabla^2\phi = 0}$$

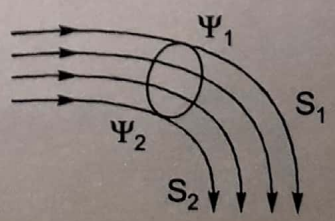
- ☑ If velocity potential function satisfies Laplace equation then it also satisfies continuity equation and hence the **flow** is **possible**.

13. Streamline Function (ψ) : Defined only for 2D, Steady and incompressible flow.

$$v = \frac{\partial\psi}{\partial x}, \quad u = -\frac{\partial\psi}{\partial y}$$

- If Ψ exists, then it automatically satisfies continuity equation and flow can be rotational or irrotational.
- If stream function satisfies Laplace equation then it is case of **Irrotational flow.** $\boxed{\nabla^2\psi = 0}$
- Discharge per **unit** length $= \psi_1 - \psi_2$

  $\psi_1 > \psi_2$



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14. Cauchy-Reimann equation

$$-\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} = u$$

$$-\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} = v$$

$$\psi = \int -u.dy + \int v(\text{terms not containing } y).dx$$

$$\phi = \int -u.dx - \int v(\text{terms not containing } x).dy$$

Remember:

☑ Equipotential lines and constant stream function lines are **orthogonal**.

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